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Équations de réaction-diffusion non-linéaires et modélisation en écologie

THÈSE

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par

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Introduction Générale

L'étude de problèmes de réaction-diffusion non-linéaires constitue l'objet de cette thèse. Elle se divise en trois parties distinctes ; la première concerne l'analyse mathématique de modèles d'invasion biologique en environnement périodique fragmenté. La deuxième étudie un modèle mathématique issu de la théorie de la combustion. La dernière partie, enfin, aborde la modélisation de la dispersion d'un insecte invasif.

Le point commun entre ces trois parties est l'utilisation et l'étude d'équations aux dérivées partielles en rapport à la modélisation, et de la forme

$$\frac{\partial u}{\partial t} = \nabla \cdot (A(x)\nabla u) - V(x) \cdot \nabla u + f(x,u), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^N.$$

Les première et deuxième parties s'intéressent ainsi plus spécifiquement aux solutions "front progressif". Les première et troisième parties s'attachent à des phénomènes d'invasion biologique modélisés par ce type d'équation.

Les modèles étudiés dans la première partie ont été récemment introduits par Shigesada *et al.* [85] ; l'environnement y est présupposé variable périodiquement, favorable en certains endroits à la survie des espèces, et défavorable en d'autres. Les équations de réaction-diffusion traitées auront donc des termes de diffusion et de réaction dépendant périodiquement de l'espace.

Nous ferons dans un premier temps l'étude complète du problème stationnaire associé à cette équation, en donnant des conditions simples assurant l'existence et l'unicité des solutions stationnaires, puis en étudiant l'influence des hétérogénéités du milieu, en particulier la répartition et l'amplitude des zones plus ou moins favorables ou défavorables, sur les conditions d'existence de telles solutions.

Dans un second temps, nous nous intéresserons à l'existence de "fronts progressifs pulsatoires", solutions des équations de réaction-diffusion citées ci-dessus, décrivant l'invasion d'un état stationnaire 0 par un état stationnaire p, en général non constant. Nous étudierons également l'influence des hétérogénéités sur la propagation des fronts.

Dans la deuxième partie, nous nous pencherons sur l'existence de flammes "planes", dans le cas d'une réaction chimique exothermique non-réversible du type $A \to B$, avec pertes de chaleur volumétriques. L'objectif sera donc de trouver des solutions du type "front progressif" à un système de deux équations de réaction-diffusion. En utilisant des outils topologiques, nous prouverons ici, sous certaines conditions, l'existence de deux solutions distinctes, et donnerons des résultats de non-existence ainsi que des estimations sur la vitesse des flammes.

Dans la troisième partie, nous nous consacrerons à la modélisation du processus d'extension spatio-temporelle d'une nouvelle espèce d'insecte, récemment introduite au Mont-Ventoux (Sud-Est de la France) par le biais de graines de cèdre du Liban. Les déplacements de l'insecte au stade adulte seront modélisés par une équation de réaction-diffusion. Le recours à des tests numériques nous permettra de mettre en évidence l'utilité de ce modèle dans l'approfondissement de la compréhension du processus d'invasion de l'insecte, et dans la prévision de sa position pour les années à venir. Nous évoquerons également les possibilités d'adaptation du logiciel réalisé au cours de ce travail à l'étude de la propagation d'autres espèces vivantes.

1 Première partie : Modèles d'invasion biologique en environnement périodique fragmenté

Les équations de réaction-diffusion du type

$$u_t = \Delta u + f(u) \quad \text{sur } \mathbb{R}^N \tag{1}$$

ont été introduites à la fin des années 30 dans des travaux de Fisher (1937) [43] et Kolmogorov, Petrovsky et Piskunov (1938) [68], pour des modèles de génétique des populations. Les non-linéarités f considérées alors étaient du type $f(u) = u(1-u)$ (loi logistique) ou ses extensions (ex. : $f(u) = u(1-u^2)$).

Ces équations, avec de telles non-linéarités, se retrouvent également en écologie et en combustion. Elles permettent de rendre compte de la propagation de fronts, solutions de la forme $u(t,x) = U(x\cdot e - ct)$, où e est la direction de propagation et c la vitesse d'invasion de l'état constant 0 par l'état 1 ($U(-\infty) = 1$ et $U(+\infty = 0)$). De tels fronts existent pour toute vitesse $c \geq c^* = 2\sqrt{f'(0)}$ [5, 68]. De plus, toute solution positive qui, à l'instant initial, est non nulle et à support compact, converge vers 1 en temps grand, en s'étendant avec la vitesse asymptotique c^*.

Si le cas de l'équation homogène (1), où les coefficients de diffusion et de réaction sont constants en espace, a donné lieu à de nombreuses études, la question de la propagation de fronts dans le cas des extensions hétérogènes de (1) n'a été envisagée que récemment. Ici les coefficients de diffusion et de réaction varient avec les hétérogénéités du milieu.

Dans ce cas, l'un des modèles les plus simples est le "patch model", dans lequel l'environnement est supposé homogène par morceaux, induisant une équation à coefficients constants par morceaux. Ce type de modèle a été introduit par Shigesada, Kawasaki et Teramoto [86] pour étudier des phénomènes d'invasion biologique dans des environnements périodiques, comme décrit dans le livre [85].

Nous nous intéressons ici aux modèles plus généraux d'invasion biologique en environnement fragmenté, représentés par les équations de réaction-diffusion

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \mathbb{R}^N \tag{2}$$

où le champ de matrices $A(x)$ et le terme non linéaire $f(x,s)$ dépendent des variables $x = (x_1, \cdots, x_N)$ de façon périodique. Dans ces modèles d'écologie proposés par Shigesada *et al.* [85], l'environnement varie périodiquement comme souligné précédemment : plus $f_u(x,0)$ est négatif, plus le milieu est défavorable à la survie des espèces. L'exemple le plus typique, utilisé dans [66, 85, 86] est $f(x,u) = u(\mu(x) - \nu(x)u)$, ou, plus simplement, $f(x,u) = u(\mu(x) - u)$.

Ce travail a pour but l'analyse mathématique rigoureuse et complète du problème (2) et des questions de propagation dans un environnement général périodique. Il s'attache en particulier à l'influence de la fragmentation et de l'amplitude des zones plus ou moins favorables ou défavorables sur les conditions de survie et de propagation d'une espèce.

A notre connaissance, et même dans le cas simplifié du "patch model", nous apportons ici la première preuve rigoureuse des résultats énoncés. Si d'autres auteurs ont obtenu des résultats concernant le problème (2), aucun en effet n'en a fait une étude complète. En outre, l'intérêt porté au cas général de l'équation (2), où les coefficients ne sont pas nécessairement constants par morceaux, où $f_u(x,u)$ peut changer de signe, et où la dimension N est quelconque, constitue une spécificité de ce travail. Ainsi, l'influence de la position des zones favorables et défavorables sur la survie des espèces a été étudiée dans le cas borné, et pour des fonctions continues par morceaux par Cantrell et Cosner [28] ; l'existence de "front progressifs pulsatoires" a été montrée, en dimension 1, et sans prouver l'existence d'une vitesse minimale par Hudson et Zinner [56] ; enfin, une formule pour la vitesse minimale des fronts c^* peut être déduite d'un travail de Weinberger [92], avec une approche différente, où il est notamment supposé *a priori* que les fronts sont croissants en temps.

1.1 Influence des hétérogénéités du milieu sur la survie des espèces

Dans ce chapitre, nous nous intéressons à l'existence et à l'unicité des états stationnaires de l'équation (2). L'état 0 étant toujours un état stationnaire, nous cherchons des solutions p du problème

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) = f(x,p) \ \text{ dans } \mathbb{R}^N, \\ p(x) > 0, \ x \in \mathbb{R}^N. \end{cases} \tag{3}$$

Sous certaines hypothèses, nous prouvons que les solutions de (3) sont en fait périodiques (L_1-périodicité en x_1,..., L_N-périodicité en x_N). Nous établissons ici une condition nécessaire et suffisante simple pour obtenir l'existence d'une solution de (3).

Nous démontrons également un nouveau résultat d'unicité du type Liouville, d'un intérêt pouvant dépasser le cadre de ce travail.

En termes d'écologie, l'existence de solutions au problème (3) correspond à la condition de survie de l'espèce. Nous obtenons en effet une description complète du comportement asymptotique des solutions de l'équation (2), et prouvons, sous certaines hypothèses, que si une solution de (3) existe, alors les solutions de (2) convergent asymptotiquement vers cette solution ; au contraire, si le problème (3) n'a pas de solution, on démontre que les solutions de (2) convergent vers 0, et donc qu'il y a extinction de l'espèce.

Ensuite, nous nous intéressons au rôle de la fragmentation de l'environnement. Ainsi, dans des situations englobant le "patch model", nous trouvons un réarrangement de l'environnement assurant toujours de meilleures chances de survie que l'environnement initial. De plus, les méthodes introduites, utilisant différents types de réarrangements, permettent d'étendre les résultats de Cantrell et Cosner [28] obtenus sur des domaines bornés, à des dimensions supérieures et des non-linéarités plus générales.

Nous nous penchons aussi sur les effets des grandes amplitudes de la non-linéarité.

Précisons que tous ces résultats peuvent être utilisés pour améliorer les chances de survie d'une espèce, mais peuvent également être vus en terme de lutte biologique.

1.1.1 Résultats : Existence, unicité, comportement en temps long et effet des hétérogénéités

L'équation considérée ici est donc (2), dont nous cherchons les solutions stationnaires, qui vérifient (3).

Le champ de matrices A et le terme non linéaire $f(\cdot, s)$ sont supposés périodiques. Rappelons qu'étant donnés $L_1, \ldots, L_N > 0$, une fonction g définie dans $\mathbb{R}^N$ est dite périodique si $g(x+k) = g(x)$ pour tout $x \in \mathbb{R}^N$ et $k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$. Soit C la cellule de périodicité $C = (0, L_1) \times \cdots \times (0, L_N)$.

On suppose qu'il existe $\nu > 0$ tel que $A(x) \geq \nu I$ pour tout x, au sens des matrices symétriques (I est la matrice identité). Enfin, A est de classe $C^{2,\alpha}$ ($\alpha > 0$), et f est de classe $C^{1,\alpha}$ en (x, u), C^2 en u. On suppose que $f(x, 0) = 0$ pour tout $x \in \mathbb{R}^N$. Nous ferons éventuellement les hypothèses suivantes :

$$\forall x \in \mathbb{R}^N, \quad s \mapsto f(x, s)/s \text{ est strictement décroissante par rapport à } s > 0 \tag{4}$$

et/ou

$$\exists M \geq 0, \ \forall s \geq M, \ \forall x \in \mathbb{R}^N, \quad f(x, s) \leq 0. \tag{5}$$

Soit λ_1 la valeur propre principale de l'opérateur $-\nabla \cdot (A(x)\nabla\ ) - f_u(x, 0)$ avec conditions de périodicité, c'est-à-dire l'unique réel pour lequel il existe une fonction ϕ, périodique, de classe C^2, vérifiant

$$-\nabla \cdot (A(x)\nabla\phi) - f_u(x, 0)\phi = \lambda_1\phi, \quad \phi(x) > 0 \quad \text{dans } \mathbb{R}^N.$$

Nous dirons que la solution 0 de (3) est instable si $\lambda_1 < 0$, et stable sinon. L'instabilité est vérifiée si $f_u(x, 0) \geq 0, \not\equiv 0$ (par exemple si $f'(0) > 0$ dans le cas où f ne dépend pas de x), mais cette condition n'est pas nécessaire.

Nos résultats d'existence et d'unicité sont résumés par le théorème suivant :

Théorème 1 *1) Supposons $\lambda_1 < 0$. Si f vérifie* (5)*, alors il existe une solution périodique p de* (3) *; si f vérifie* (4)*, alors il existe au plus une solution bornée de* (3)*, qui est alors périodique.*

2) Si $\lambda_1 \geq 0$ et f vérifie (4)*, alors il n'existe pas de solution bornée de* (3).

Le résultat d'existence de la partie 1) se vérifie pour des équations plus générales comprenant éventuellement des termes de transport. Nous le démontrons en adaptant la méthode classique des sur- et sous-solutions.

Le résultat d'unicité est de type Liouville et ne suppose pas a priori que les solutions sont périodiques. La recherche d'une constante strictement positive minorant les solutions constitue la partie difficile de la démonstration. Pour le prouver, les solutions u^y du problème (3) translaté dans la direction $-y$ sont comparées aux fonctions propres principales ϕ_R^y du linéarisé en 0 de ce problème sur une boule B_R. Une série de lemmes nous permet de prouver que pour R assez grand, ϕ_R^y est une sous-solution (avec une condition de normalisation bien choisie) du problème (3) translaté. Dès lors, $u^y(0) > \phi_R^y(0)$, et ensuite un argument de périodicité permet de prouver que toute solution positive bornée de (3) est minorée par une constante strictement positive. L'unicité est alors une conséquence du principe du maximum, et la périodicité une conséquence de l'unicité.

Des conditions d'existence et d'unicité des solutions p de (3), découle le comportement en temps grand des solutions $u(t,x)$ de (2) avec donnée initiale $u_0(x) \geq 0$, $u_0(x) \not\equiv 0$, bornée et uniformément continue :

Théorème 2 *Sous les hypothèses* (4) *et* (5)*, alors* $u(t,x) \to p(x)$ (*unique solution bornée de* (3)) *dans* $C^2_{loc}(\mathbb{R}^N)$ *quand* $t \to +\infty$ *si* $\lambda_1 < 0$*, et* $u(t,x) \to 0$ *uniformément en* x *quand* $t \to +\infty$ *si* $\lambda_1 \geq 0$.

Ainsi, sous les hypothèses (4-5), la condition $\lambda_1 < 0$ est équivalente à la survie des espèces (avec une concentration finale obtenue égale à l'unique solution bornée p de (3)), tandis qu'il y a extinction si $\lambda_1 \geq 0$.

Le signe de la valeur propre λ_1, qui ne dépend que de A et de $f_u(\cdot,0)$, joue donc un rôle crucial, et nous précisons dans la suite de cette section l'influence des hétérogénéités du milieu sur la valeur de λ_1, que nous notons $\lambda_1 = \lambda_1[\mu]$ avec $\mu(x) = f_u(x,0)$.

Théorème 3 *1) On a* $\lambda_1[\mu] \leq \lambda_1[\mu_0]$*, où* $\displaystyle\int_C \mu = \mu_0|C|$ *et* $|C|$ *est la mesure de Lebesgue de* C.

2) Si $A = I$*, alors* $\lambda_1[\mu^*] \leq \lambda_1[\mu]$*, où* μ^* *est la symétrisation de Steiner périodique de* μ.

3) Si $\displaystyle\int_C \mu \geq 0$ *et* $\max_{\mathbb{R}^N} \mu > 0$*, alors* $\lambda_1[B\mu] < 0$ *pour tout* $B > 0$*, et* $\lambda_1[B\mu]$ *est strictement décroissant par rapport à* $B \geq 0$*. Si* $\displaystyle\int_C \mu < 0$*, alors* $\lambda_1[B\mu] > 0$ *pour* $B > 0$ *petit. Si* $\max_{\mathbb{R}^N} \mu > 0$*, alors* $\lambda_1[B\mu] < 0$ *au moins pour* $B > 0$ *suffisamment grand et* $\lambda_1[B\mu]$ *est strictement décroissant en* B *dès que* $\lambda_1[B\mu] < 0$.

Ces trois résultats sont prouvés en utilisant une caractérisation variationnelle de $\lambda_1[\mu]$. Leur originalité tient en partie à l'utilisation d'une formule de B. Kawohl [65], nous permettant d'affirmer que $\displaystyle\int_C |\nabla\phi|^2 \geq \int_C |\nabla\phi^*|^2$, où ϕ est la fonction propre principale associée à $\lambda_1[\mu]$.

A la lumière des théorèmes 2 et 3, ces résultats ont une interprétation simple, mais non triviale intuitivement, en termes de survie de l'espèce. Ainsi, la partie 1) signifie que, à moyenne égale, un taux de natalité hétérogène (non constant en x) favorise la survie des espèces, et ce quel que soit le type de diffusion. Pour préciser le résultat 2), nous devons rappeler les notions de symétrisé périodique de Schwarz et de Steiner.

En dimension 1, le réarrangement périodique de Schwarz $\mu^*(x)$ est l'unique fonction L-périodique sur $\mathbb{R}$, symétrique par rapport à la droite $x = L/2$, décroissante sur $[0, L/2]$, puis croissante sur $[L/2, L]$ et ayant la même distribution que μ. En dimension N, on appelle réarrangement périodique de Steiner, dans l'ordre $x_1, \cdots, x_N$, la fonction μ^* obtenue en faisant successivement les réarrangements périodiques de Schwarz par rapport à chacune des variables $x_1, x_2, \ldots, x_N$.

La figure 1 ci-dessous donne un exemple de réarrangement périodique de Steiner (dans $\mathbb{R}^2$) dans les directions x puis y, et illustre le mode de regroupement des zones défavorables.

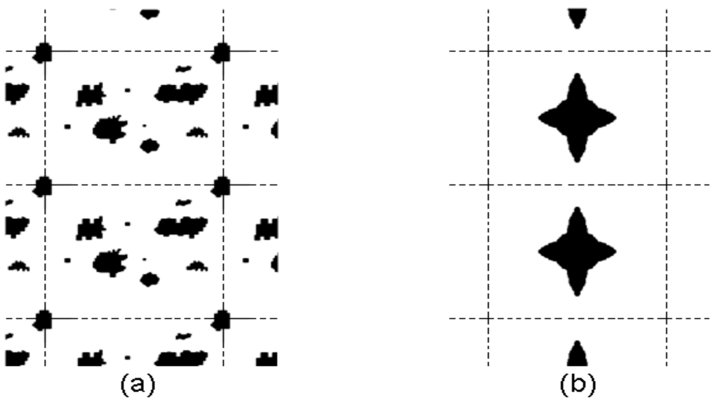

FIG. 1 – Les effets d'un réarrangement périodique de Steiner sur les zones défavorables. Répartition des zones défavorables : (a) pour $\mu(x)$; (b) pour $\mu^*(x)$.

Le théorème 3, 2), montre que la configuration μ^* (concentration des zones défavorables) donne plus de chances de survie à l'espèce. Cela implique également que la répartition optimale des zones est nécessairement stable par réarrangement de Steiner (quel que soit l'ordre des variables dans lequel on fait le réarrangement). Cependant, comme nous le soulignerons dans la section 4 de cette introduction, la recherche de cette répartition optimale est un problème ouvert.

D'après la partie 3), dans le cas d'un milieu qui n'est pas complètement défavorable, l'amplification suffisamment grande du taux de natalité est meilleure pour la survie des espèces.

Dans le cas des domaines bornés, avec conditions de Dirichlet ou Neumann, nous élargissons les résultats de Cantrell et Cosner [28], à des non-linéarités plus générales, et à des dimensions supérieures. Nous considérons en effet une non-linéarité similaire à celle du cas périodique, et apportons la preuve, dans le cas Dirichlet, qu'un réarrangement de Steiner de μ donne toujours de meilleures chances de survie que μ en termes de λ_1. Dans le cas avec conditions de Neumann, on obtient un résultat similaire au cas périodique, en utilisant un réarrangement monotone de Steiner, faisant intervenir une inégalité démontrée dans [12] (cf. partie I, chapitre 1, section 6).

1.2 Invasions biologiques et fronts progressifs pulsatoires

Dans ce second chapitre, nous examinons les phénomènes de propagation, et en particulier la propagation de fronts, associés à l'équation (2).

Nous établissons le lien entre les conditions de survie des espèces (existence d'une solution $p > 0$ de (3)), et les conditions de propagation de fronts progressifs pulsatoires (voir définition ci-après), où l'état 0 est envahi par l'état hétérogène p.

Nous prouvons en premier lieu l'existence d'une vitesse c^* telle qu'une solution du type front progressif pulsatoire existe si et seulement si $c \geq c^*$. Nous donnons ensuite une formule variationnelle pour c^*, et analysons les effets de l'hétérogénéité du milieu sur la vitesse de propagation. Nous démontrons en particulier une dépendance croissante entre la vitesse d'invasion et l'amplitude du taux de natalité.

1.2.1 Résultats : Existence d'une vitesse minimale c^*, croissance des solutions, caractérisation de c^* et dépendance de c^* par rapport à la non-linéarité f

Etant donnés une solution p de (3) et un vecteur normé e de $\mathbb{R}^N$, on appelle front pulsatoire se propageant dans la direction e avec la vitesse effective c ($\neq 0$), une solution $u = u(t,x)$ ($t \in \mathbb{R}$, $x \in \mathbb{R}^N$) de (2) telle que

$$\begin{cases} \forall\, k \in \prod_{i=1}^{N} L_i\mathbb{Z},\ \forall\, x \in \mathbb{R}^N,\ u\left(t - \dfrac{k\cdot e}{c}, x\right) = u(t, x+k), \\ u(t,x) \underset{x\cdot e \to +\infty}{\to} 0,\ u(t,x) - p(x) \underset{x\cdot e \to -\infty}{\to} 0, \end{cases} \tag{6}$$

où les limites sont prises localement en t, et uniformément dans les directions orthogonales à e. Cette notion de front pulsatoire généralise celle de front progressif du type $u(t,x) = U(x \cdot e - ct)$ pour l'équation homogène (1).

Théorème 4 *Supposons que f vérifie (4-5) et que $\lambda_1 < 0$. Soit p l'unique solution de (3). Il existe $c^* > 0$ tel que l'équation (2) avec les conditions (6) a une solution (c,u) si et seulement si $c \geq c^*$. De plus, toute solution u est strictement croissante en t. Enfin, c^* est donnée par $c^* = \min_{\lambda>0} k(\lambda)/\lambda$, où $k(\lambda)$ est la valeur propre de l'opérateur L_λ défini par*

$$L_\lambda \psi = \nabla\cdot(A(x)\nabla\psi) - 2\lambda\, eA(x)\nabla\psi + [-\lambda\nabla\cdot(A(x)e) + \lambda^2 eA(x)e + f_u(x,0)]\psi$$

avec conditions de périodicité.

Pour prouver le résultat d'existence, nous nous sommes inspirés des méthodes de H. Berestycki et F. Hamel [13]. Cependant, ici, la non-linéarité f n'est plus nécessairement positive, ce qui induit une nouvelle difficulté. La méthode de régularisation et d'approximation des non-linéarités par cut-off ne pouvant plus s'appliquer, nous proposons une autre méthode. Considérant des problèmes régularisés dans des domaines bornés, nous prouvons cette fois directement que l'ensemble des vitesses pour lesquelles une solution existe est un intervalle, et c^* est défini comme la borne inférieure de cet intervalle. Dans la démonstration, nous faisons notamment plusieurs fois appel à une propriété sur le signe de l'énergie de l'état stationnaire positif p, solution de (3). Cette propriété est établie dans le chapitre 1, au cours de l'étude du problème stationnaire (3).

Sous les hypothèses du théorème 4, c^* ne dépend que de A, e et $f_u(x,0)$. Nous supposons que A et e sont fixés, et que $f_u(x,0) = \mu(x)$, où μ est une fonction périodique donnée et $B > 0$ un paramètre réel. Notons $c^* = c^*[\mu]$.

Théorème 5 *Supposons que f vérifie (4-5) et que A est constante.*

1) Si $\overline{m} = \int_C \mu \ / \ |C| > 0$, alors $c^*[\mu] \geq c^*[\overline{m}] = 2\sqrt{eAe \ \overline{m}}$.

2) Si $\overline{m} \geq 0$ et $M = \max_{\mathbb{R}^N} \mu > 0$, alors $c^*[B\mu]$ est strictement croissante en $B > 0$. De plus,

$$2\sqrt{eAe \ B \ \overline{m}} \leq c^*[B\mu] \leq 2\sqrt{eAe \ B \ M} \quad \textit{pour tout } B > 0,$$

$$\frac{1}{2}\sqrt{eAe \ M} \leq \liminf_{B\to+\infty} \frac{c^*[B\mu]}{\sqrt{B}} \leq \limsup_{B\to+\infty} \frac{c^*[B\mu]}{\sqrt{B}} \leq 2\sqrt{eAe \ M},$$

et $c^*[B\mu]/\sqrt{B} \to 2\sqrt{eAe \ \overline{m}}$ quand $B \to 0^+$.

Pour établir ces résultats, on utilise la caractérisation de c^* suivante :

$$c^* = \min_{\lambda>0} \frac{-k_\lambda(B)}{\lambda}, \tag{7}$$

où $k_\lambda(B)$ est la valeur propre principale de l'opérateur

$$-\mathcal{L}_{B,\lambda}\phi = -\nabla\cdot(A\nabla\phi) - 2\lambda Ae\cdot\nabla\phi - \lambda^2 \ eAe\phi - B\mu(x)\phi,$$

avec conditions de périodicité. En effet, cette caractérisation est équivalente à la formule obtenue dans le théorème 4 (cf. [13] et [93]). Les résultats énoncés dans le théorème 5 découlent des propriétés de $k_\lambda(B)$.

D'après ce qui vient d'être établi, à moyenne constante, la présence d'hétérogénéités dans l'environnement (taux de natalité non constant) augmente ainsi la vitesse des fronts par rapport au cas homogène. De plus, cette vitesse augmente avec l'amplitude B du taux de natalité et se comporte en $\sqrt{B}$ pour B grand et petit. Lorsque le milieu est très différencié (B grand), la vitesse des fronts devient de plus en plus grande, même si l'environnement est ainsi parfois très défavorable à certains endroits .

2 Deuxième partie : Etude d'un modèle de combustion avec pertes de chaleur

Cette partie concerne l'existence de solutions du type front progressif pour un modèle de combustion. Rappelons que les problèmes de flammes planes se propageant dans des gaz prémélangés sont décrits par des systèmes scalaires d'équations de réaction-diffusion non-linéaires couplées. La plupart des aspects de la propagation de flammes peuvent être étudiés dans le cas du modèle simplifié d'une réaction chimique exothermique non-réversible du type $A \to B$. Dans ce cas, le problème associé se réduit à un système de deux équations de réaction-diffusion couplées, où les inconnues sont i) la température u du mélange, ii) la concentration v en réactant et iii) la vitesse c de la flamme. Ce type de problème, considéré dans un cylindre infini, s'écrit :

$$\left\{ \begin{array}{ll} -u'' + cu' & = f(u,v) \\ -\Lambda v'' + cv' & = g(u,v) \end{array} \right. \quad \text{sur } \mathbb{R}, \tag{8}$$

avec des conditions aux limites en $\pm\infty$. Il faut souligner qu'une température d'ignition peut exister (par exemple pour certains hydrocarbures), de telle sorte, que pour des températures trop basses, la réaction n'a pas lieu (ce qui impose une condition sur $f(\cdot, v)$ et $g(\cdot, v)$).

De nombreux travaux se sont intéressés au système (8), avec température d'ignition, en particulier dans le cas adiabatique, où les pertes de chaleur sont négligées [23], [46], [72], [26]. Ainsi, avec $g(u,v) = -f(u,v)$, certains calculs se trouvaient simplifiés.

Nous nous intéressons au cas avec pertes de chaleur ; le système de réaction-diffusion associé est alors

$$\left\{ \begin{array}{ll} -u'' + cu' & = f(u,v) - \lambda h(u) \\ -\Lambda v'' + cv' & = -f(u,v) \end{array} \right. \quad \text{sur } \mathbb{R}, \tag{9}$$

avec les conditions aux limites

$$\left\{ \begin{array}{ll} u(-\infty) = 0, & u(+\infty) = 0, \\ v(-\infty) = 1, & v'(+\infty) = 0. \end{array} \right. \tag{10}$$

Dans ce cas, Zeldovitch a prouvé, dans des travaux précurseurs [95] (1941), et en utilisant des méthodes asymptotiques, que ce système avait deux solutions du type front progressif, dans le cas limite des hautes énergies d'activation ($f(u,v) = \dfrac{1}{\varepsilon^2} exp\left(\dfrac{u-1}{\varepsilon}\right)\chi(u)v$, $\varepsilon \to 0$). Plus récemment, Glangetas et Roquejoffre, dans [47], ont retrouvé ce résultat comme une conséquence d'une formule obtenue par Joulin et Clavin [60], [61], qu'ils ont à leur tour rigoureusement prouvée dans [47]. Des non-linéarités plus générales ont été étudiées par Giovangigli dans [46], où il démontre l'existence d'une solution pour une vitesse c fixée (considérant le paramètre de pertes de chaleur λ comme une inconnue du problème), avec un nombre de Lewis $Le = 1$ (remarquons que $Le = 1/\Lambda$). Ici, nous restons dans le cadre des articles de Berestycki, Nicolaenko et Scheurer [23], en conservant c comme une inconnue du problème, et établissons l'existence de deux solutions distinctes pour les petites valeurs de λ. De plus, nous prouvons ce résultat quelle que soit la valeur du nombre de Lewis Le, et obtenons ainsi un résultat de non-unicité.

En effet, Marion [72] a montré l'unicité dans le cas adiabatique, pour des nombres de Lewis plus grands que 1, et Bonnet, quant à lui, a mis en évidence, dans [26], des cas où l'unicité ne se vérifie pas pour des nombres de Lewis plus petits que 1. Nous démontrons ici qu'il n'y a jamais unicité dans le cas non-adiabatique (pour de faibles pertes de chaleur λ).

De plus, nous prouvons, dans certains cas, que la solution avec la plus grande vitesse de propagation (sur les deux solutions obtenues) converge vers la solution du problème adiabatique quand $\lambda \to 0$, alors que l'autre vitesse converge vers 0.

Nous établissons également de nouvelles bornes pour les solutions du problème (9-10). Dans ses travaux, Giovangigli avait prouvé (dans [46]) avec $Le = 1$, que la vitesse c était bornée par la vitesse c_{ad} solution du problème adiabatique. Ici, nous obtenons que, pour $Le \leq 1$ (rappelons que $Le = 0.4$ pour l'hydrogène), c est inférieure à la vitesse solution d'un problème adiabatique scalaire. De plus, pour tout $Le > 0$, nous donnons une majoration explicite de c, qui ne dépend pas de λ. Nous prouvons également que la réaction n'est pas totale, et calculons une borne inférieure pour la quantité de réactant non brûlée.

2.1 Résultats : Existence de deux solutions, estimations sur les solutions

Les hypothèses sur la non-linéarité f sont les suivantes : il existe deux fonctions p et g telles que

$$f(u,v) = p(u)g(v) \text{ sur } \mathbb{R} \times \mathbb{R}, \tag{11}$$

où la fonction p est Lipschitzienne sur $\mathbb{R}$, croissante, et admet une température d'ignition θ :

$$\exists\, \theta \in (0,1) \text{ tq. } p(x) = 0 \text{ si } x \leq \theta \text{ et } p(x) > 0 \text{ si } x > \theta, \tag{12}$$

et la fonction g est continue, strictement croissante sur $\mathbb{R}_+$ et telle que

$$g < 0 \text{ sur } \mathbb{R}_-^* \text{ et } \; g(0) = 0; \tag{13}$$

De plus, nous sommes amenés à faire une hypothèse technique sur la fonction g, vérifiée par exemple pour des fonctions du type $g(y) = y^n$, avec $n > 0$, ou encore pour des fonctions g, n fois dérivables, et telles que la n^{ieme} dérivée en 0 soit non nulle ($n \geq 1$).

Nous supposons que la fonction h est bornée dans C^1, strictement croissante, et satisfait $h(0) = 0$.

Soient (u_{ad}, v_{ad}, c_{ad}) les solutions du problème sans pertes de chaleur ((9) avec $\lambda = 0$, voir [23] pour leur existence), et (u_s, c_s) l'unique solution (voir [23]) du problème adiabatique suivant

$$-\Lambda u_s'' + c_s u_s' = f(u_s, 1 - u_s), \tag{14}$$

avec les conditions aux limites

$$u_s(-\infty) = 0, \; u_s(+\infty) = 1. \tag{15}$$

Nous démontrons alors que

Théorème 6 *1) Pour tout $\Lambda > 0$, et pour $\lambda > 0$ assez petit, il existe deux solutions distinctes et non triviales (u_1, v_1, c_1) et (u_2, v_2, c_2) au problème (9 -10), avec $c_1 < c_2$.*

2)Pour tout $\Lambda > 0$, $c_1 \to 0$ quand $\lambda \to 0$. De plus, si $\Lambda \leq 1$ et $g(y) = y$, (u_2, v_2, c_2) converge sur tout compact vers (u_{ad}, v_{ad}, c_{ad}) quand $\lambda \to 0$. Le même résultat est vrai pour le terme de réaction plus général $f(x, y)$ dans le cas $\Lambda = 1$.

Pour obtenir le résultat d'existence, nous étudions d'abord le problème sur un ouvert borné, afin d'utiliser, comme dans [23], [46] et [72] un argument de degré topologique de Leray-Schauder (voir [79]). Cependant, après avoir fait les estimations *a priori* et avoir calculé le degré, nous constatons que ce dernier est nul (à cause du nombre pair de solutions), ce qui a conduit Giovangigli à changer le rôle de c et de λ. Pour pouvoir, malgré tout, prouver l'existence de solutions, on doit donc obtenir une nouvelle estimation *a priori*, nous permettant de séparer deux types de solutions, certaines avec une vitesse c proche de 0, et d'autres avec une vitesse c plus grande. Cette estimation est obtenue en faisant tendre λ vers 0. A la difficulté mentionnée ci-dessus s'ajoute celle de la non-unicité des solutions avec $\lambda = 0$ pour certaines valeurs de Λ (cf. [26]), qui est résolue par une succession de lemmes techniques. Suite au calcul du degré sur deux ouverts distincts, nous obtenons le résultat, après passage à la limite. La partie 2) du théorème 6 se déduit des calculs effectués pour obtenir le résultat d'existence, et des résultats d'unicité de Marion [72].

Les solutions obtenues ci-dessus ne sont pas nécessairement les seules, et nous n'avons prouvé leur existence que dans le cas de petites valeurs de λ. Les résultats présentés dans le théorème suivant sont vrais pour toutes les solutions (u, v, c) du problème (9 -10).

Théorème 7 *1) Si* $\lambda > \dfrac{f(1,1)}{h(\theta)}$, *le problème (9-10) n'a pas de solution.*

2) Si g est $K-$lipschitzienne et si (u, v, c) n'est pas triviale, on a

$$v(+\infty) > exp\left(-\frac{Kp(1)}{\lambda h(\theta)}\right).$$

3) Pour tout $\Lambda \geq 1$, $0 < c \leq c_s$, où (u_s, c_s) est la solution de (14-15).

4) Soit $\sigma_1 = \max\limits_{s\in(\theta,1)} \dfrac{f(s, 1-s)}{s}$ *et* $\sigma_2 = \max\limits_{s\in[0,1]} f(1 - \min\{1, \Lambda\}s, s)$, *alors*

$$0 < c < 2\sqrt{\sigma_1 \Lambda} \text{ pour tout } \Lambda \geq 1 \text{ et } 0 < c < \sqrt{\frac{\sigma_2}{\theta}} \text{ pour tout } \Lambda > 0.$$

Le premier résultat est donc un résultat de non-existence, déjà donné par Giovangigli [46] dans le cas $\Lambda = 1$. Il laisse supposer l'existence d'un λ^* critique tel que, pour tout $\lambda < \lambda^*$, il existe deux solutions, et tel que, pour tout $\lambda > \lambda^*$, il n'y a pas de solution. Nous reviendrons sur cette question dans notre section 4.

Le deuxième résultat donne une borne inférieure pour les gaz non brûlés. Nous avons dû supposer que la fonction g était lipschitzienne ; en effet, le théorème de Cauchy-Lipschitz nous permet d'établir que $v(+\infty) \neq 0$; nous divisons ensuite par v l'équation satisfaite par v, et l'intégrons par parties pour obtenir le résultat.

Le troisième résultat et la première estimation du 4) sont obtenus en s'inspirant de calculs issus de [23], [72] ou encore d'un principe de comparaison prouvé dans [20]. La fonction u n'étant plus croissante dans le cas non-adiabatique, les mêmes méthodes ne peuvent pas s'appliquer directement. Nous contournons cette difficulté en faisant l'analogie entre la fonction u du cas adiabatique et la fonction $1-v$ dans le cas avec pertes de chaleur, et en introduisant la fonction $j(y) = -v' \circ v^{-1}(1-y)$.

Notons que la deuxième estimation du 4) s'obtient très facilement par un principe du maximum.

3 Troisième partie : Modélisation de la dispersion d'un insecte invasif

Ce volet correspond à la partie la plus "appliquée" de cette thèse. Il est en effet consacré à la réalisation d'un modèle, à la résolution numérique du problème de réaction-diffusion associé, et à la conception d'un logiciel permettant d'utiliser ce modèle. Il s'intègre dans le cadre du projet "Caractérisation des potentialités invasives d'un nouveau ravageur des graines de cèdre récemment introduit au Mont-Ventoux", conclu entre l'INRA et le Ministère de l'Agriculture. Ce projet a déjà donné lieu à la publication d'un article biologique [40].

L'insecte étudié ici est *Megastigmus schimitscheki* Novitzky. D'autres espèces de *Megastigmus*, déjà présentes en France, ont fait l'objet de modèles, mais sur des zones moins étendues (étude de l'impact d'un *Megastigmus* sur la production de graines d'un verger, [59]). Ici, nous nous intéressons à un insecte dont la date et le lieu d'introduction (Mont-Ventoux, 1994) sont connus, ce qui facilite l'étude des processus de dispersion à partir du point source, au contraire d'espèces plus anciennes et maintenant réparties sur l'ensemble du territoire. En outre, l'insecte étudié étant invasif, il n'a pas de prédateur ; remarquons que le cèdre est lui-même une essence introduite, et n'a de ce fait également que très peu de prédateurs. Ces trois points ont beaucoup contribué à la possibilité de mise en place d'un modèle simple et réaliste.

Une étude antérieure [40], menée dans le cadre de ce projet, nous fournit des données sur la biologie de l'insecte.

Grâce à la synthèse des caractéristiques de l'insecte, qui nous a permis d'obtenir une formulation plus claire du problème, nous avons construit un modèle divisé en deux parties. Dans la première, nous simulons la phase ovo-larvaire ; dans la seconde, nous calculons la dispersion des adultes en utilisant une équation de réaction-diffusion :

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) - v_* V(x,t) \cdot \nabla u(t,x) + f(t,x) - X(t,x)u, \tag{16}$$

pour $t \in [0, N_j]$ (où N_j est le nombre de jours total où des adultes sont présents), et $x \in \mathbb{R}^2$. Le terme $f(t,x)$ modélise l'émergence progressive des insectes, tandis que $X(t,x)$ correspond au taux de mortalité quotidien pendant la période de vol. Le coefficient de diffusion D est constant, alors que le terme de transport est variable en temps et en espace (des modèles pour d'autres types d'insectes se sont montrés efficaces avec des valeurs de transport variables et un terme de diffusion fixe, [9]). Concernant la variation de V en espace, nous tenons compte de la distance de la forêt la plus proche. Nous supposons en effet que l'insecte ne dispose pas de critères "décisionnels" lui commandant de sortir de la forêt, qu'il n'y parvient que par un mécanisme de diffusion aléatoire, et s'élève ensuite pour être davantage soumis au vent, en fonction de son éloignement de la forêt. Pour préciser la force du vent à laquelle est soumis l'insecte, nous utilisons une formule de [74], utilisée pour calculer la force du vent à laquelle est soumise une graine pour une végétation rase. En effet, l'insecte s'élevant suffisamment haut, nous pouvons supposer qu'à l'extérieur de la forêt de cèdre le milieu correspond plus à une végétation “rase” qu'à une forêt dense :

$$V(x,t) = V_e(x,t)\min\{ln[d(x, foret)(e-1)/d_a + 1], 1\},$$

où V_e est le vent enregistré et $d(x, foret)$ la distance jusqu'à la forêt de cèdre la plus proche.

Les modèles de réaction-diffusion sont souvent utilisés en dynamique des populations, en particulier dans les cas où l'on ne peut suivre les individus pour observer leurs mouvements (voir les livres de Turchin [89] et de Shigesada & Kawasaki [85]).

Nous résolvons cette équation en utilisant une méthode de directions alternées (à pas fractionnaires), proposée par Godunov [7] et Yanenko [94] pour l'équation de la chaleur.

Nous estimons ensuite les paramètres D et v_*. Pour D, nous utilisons une formule donnée dans le livre de Shigesada et Kawasaki [85], nous permettant de déduire la valeur de la diffusion à partir des résultats expérimentaux dont nous disposons. Pour le calcul de v_*, nous utilisons une méthode générale décrite par Turchin (d'après des travaux de Banks et al. [8]). Cette étape consiste simplement, après avoir fait des hypothèses initiales sur la valeur du paramètre à estimer, à calculer les prédictions du modèle, et à optimiser la valeur du paramètre pour ajuster ces prédictions aux valeurs expérimentales.

Nous effectuons ensuite différents tests numériques pour vérifier la validité du modèle et quantifier l'influence de certains traits adaptatifs et de certaines configurations de terrain sur la dispersion de l'insecte. Les simulations rendent compte de résultats proches des mesures effectuées sur le terrain. De plus, d'un point de vue qualitatif, le comportement de l'insecte, tel que décrit par ces tests, est cohérent avec les attentes des biologistes. Ces calculs numériques nous ont également permis de mieux connaître le rôle la diapause prolongée (émergence en retard d'un ou deux ans), et d'avoir une meilleure estimation de son taux. Ils nous ont en outre donné des informations sur l'influence de la longueur de la période d'émergence ou encore la présence d'arbres isolés sur la dispersion des insectes.

Nous évoquons finalement les limites de ce type de modèle dans le cas où les phénomènes météorologiques ont une influence majeure sur la dispersion de l'insecte. En effet, l'impossibilité de les prévoir d'une année sur l'autre réduit les possibilités de prévisions de la dispersion à long terme. C'est dans cette optique que nous avons développé un logiciel (avec le langage de programmation Matlab®) s'adaptant à d'autres configurations de terrain et d'autres espèces vivantes, dont on pourrait prédire l'évolution sur de nombreuses années de façon plus fiable que dans le cas traité ici. Nous pensons notamment à des espèces dépourvues d'ailes, ou pourvues d'ailes suffisamment puissantes pour que l'influence du vent sur leur mouvement soit négligeable. Le terme de transport serait alors moins variable en temps (bassin d'attraction par exemple, ou pente pour des végétaux).

4 Problèmes ouverts et perspectives

Partie 1 : nous avons prouvé qu'un réarrangement périodique de Steiner des zones favorables et défavorables d'un environnement périodique offrait toujours de meilleures chances de survie à une espèce que la configuration initiale. Cela prouve que la configuration optimale de l'environnement en termes de survie est stable par symétrie de Steiner. La recherche d'une telle configuration reste cependant un problème ouvert.

Partie 2 : nous avons démontré l'existence de deux solutions "front progressif" pour de faibles valeurs du paramètre de pertes de chaleur λ, et leur nonexitence pour les grandes valeurs de ce paramètre. Un problème ouvert naturel est donc de déterminer une valeur "critique" λ^* du paramètre de pertes de chaleur, tel que le problème considéré admette une solution si et seulement si $\lambda \leq \lambda^*$.

Partie 3 : une adaptation du modèle construit est prévue, dans le cadre d'un autre projet avec l'INRA, à la modélisation de l'expansion spatiale d'insectes forestiers en relation avec les changements climatiques.

Partie I : Analysis of the periodically fragmented environment model

Note au CRAS

Chapitre 1 : Influence of periodic heterogeneous environment on species persistence

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Sommaire

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1 Introduction

"In the last two decades, it has become increasingly clear that the spatial dimension and, in particular, the interplay between environmental heterogeneity and individual movement, is an extremely important aspect of ecological dynamics."

P. Turchin, *Qualitative Analysis of Movement*[1]

Reaction-diffusion equations of the type

$$u_t = \Delta u + f(u) \text{ in } \mathbb{R}^N \tag{1}$$

have been introduced in the celebrated articles of Fisher (1937) [43] and Kolmogorov, Petrovsky and Piskunov (1938) [68]. The initial motivation came from population genetics and the scope was to shed light on spatial spreading of advantageous genetic features. The nonlinear reaction term considered there are that of a logistic law of which the archetype is $f(u) = u(1-u)$ or extensions like $f(u) = u(1-u^2)$.

Several years later, Skellam (1951) [87] used this type of models to study biological invasions, *i.e.* spatial propagation of species. With these he succeeded to propose quantitative explanations of observations, in particular of muskrats spreading throughout Europe at the beginning of 20th Century or the early dissemination of birch trees in Great Britain.

Since these pioneering works, this type of equation has been widely used to model spatial propagation or spreading of biological species (bacteria, epidemiological agents, insects, plants, etc). Systems involving this type of equations have also been proposed to model the spread of human cultures (Compare in particular Cavalli-Sforza et al. [3, 32]).

A vast mathematical litterature has been devoted to the *homogeneous* equation (1). Of particular interest is to understand the structure of *travelling front solutions* and their stability, as well as *propagation* or spreading properties that this equation exhibits. The former are solutions of the type $u(t,x) = U(x \cdot e - ct)$ for any given direction e ($|e| = 1$, e is the direction of propagation) and $U : \mathbb{R} \to (0,1)$. The latter are related to the fact that starting with an initial datum $u_0 \geq 0$, $u_0 \not\equiv 0$ which vanishes outside some compact set, then $u(t,x) \to 1$ as $t \to +\infty$ and the set where u is, say, close to 1 expands at a certain speed which is the *asymptotic speed of spreading*. The papers of Aronson and Weinberger [5], and of Fife and McLeod [42], in particular, represent two mathematical milestones in the literature.

With the important exception of the work of Gärtner and Freidlin (1979)[44] on the asymptotic speed of spreading, it is only relatively recently that the questions of travelling fronts, propagation and spreading have been addressed within the framework of *heterogeneous* extensions of (1) (see e.g. [13, 56, 85, 93]).

In ecological modelling or for biological invasions, indeed, the heterogeneous character of the environment plays an essential role. It appears that *even at macroscopic scales*, the medium and its various characteristics are far from homogeneous. In the words of Kinezaki et al. [66] : "... natural environments are generally heterogeneous. For example, they are usually mosaic of heterogeneous habitats such as forests, plains, marshes and so on. Furthermore, they are often fragmented by natural or artificial barriers like rivers, cultivated fields and roads, etc. Thus growing attention has been paid in recent years to the question of how such environmental

[1] 1998, Sinauer Assoc. Inc., Sunderland, Mass.

fragmentations influence the spreading and persistence of invading species". For recent works on this aspect of ecological modelling, we refer the reader for instance to [54, 57, 62] ; compare also the references on this model quoted in [66].

A first approximation to heterogeneous environments, therefore, is the so-called *patch model.* In it, one assumes a mosaic of differentiated environments, each of which having a relatively well defined structure which one might consider as homogeneous. This involves an equation with piecewise constant coefficients (compare below). This type of model has been proposed by Shigesada, Kawasaki and Teramoto [86] to study biological invasions in periodic environments and is described in the book [85]. In the recent years a vast literature has been devoted to the study of the role of fragmentation of the environment on species survival. It is concerned mostly with space dimenson 1, and with bounded domains, which is different from the point of view that one has in studying spreading (see i particular [27, 28, 29].

More generally, a *periodic heterogeneous* model is proposed to investigate the effect of heterogeneity of the environment for more general periodic frameworks. Equation (1) is generalized to :

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \mathbb{R}^N, \tag{2}$$

where the diffusion matrix $A(x)$ and the reaction term $f(x,u)$ now depend on the variables $x = (x_1, \cdots, x_N)$ in a periodic fashion.

The typical example considered in [66, 85, 86] is

$$f(x,u) = u(\mu(x) - \nu(x)u), \tag{3}$$

or even, simply,

$$f(x,u) = u(\mu(x) - u). \tag{4}$$

Here, $\mu(x)$ can be interpreted as an *effective birth rate* of the population and $\nu(x)$ reflects a saturation effect related to competition for resources.

Thus, regions of space where $\mu(x)$ is relatively high represent favourable zones where the species can develop well. On the contrary, low $\mu(x)$ regions are less favourable to the species. We will consider cases where $\mu(x)$ can actually become negative. In such a region, if isolated from other regions, the species would actually die out.

The way the diffusion matrix $A(x)$ depends on x in more or less favourable environments varies from one case to another. As pointed out by Shigesada and Kawasaki [85], certain species, upon arriving in unfavourable environments, speed up (meaning, say in one dimension, that $A(x)$ increases) while the progression of others is hindered (meaning that $A(x)$ decreases).

The periodic patch model considered by Shigesada and Kawasaki is a particular important case of this periodic framework. As was already indicated, it consists in considering that the environment is made up of patches –each of which is homogeneous– arranged in a periodic fashion. For instance, in one dimension, with period L, in the patch model, one assumes that μ and A are piecewise constant, that is

$$\mu(x) = \begin{cases} \mu^+ & \text{in } E^+ \subset [0,L], \\ \mu^- & \text{in } E^- = [0,L]\backslash E^+, \end{cases}$$

and

$$A(x) = \begin{cases} a^+ & \text{in } E^+, \\ a^- & \text{in } E^-. \end{cases}$$

Then, $\mu(x)$ and $A(x)$ are extended periodically to $\mathbb{R}$. Here, μ^+, μ^-, a^+ and a^- are constants.

Saying that the environment E^+ is more favourable than E^- means that

$$\mu^+ > \mu^-.$$

As mentioned before, there is no reason to have a general inequality between a^+ and a^-.

The case of the periodic patch model above, in one dimension, is discussed in [86], relying on numerical computations and heuristic arguments.

One can also formulate similar models in higher dimensions. For instance, in dimension two, if $C = [0, L_1] \times [0, L_2]$ ($L_1,\ L_2 > 0$) then one considers the same definitions as above for problem (2) with $A(x_1, x_2)$ and $f(x_1, x_2, u)$ being L_1-periodic in x_1 and L_2-periodic in x_2, with (E^+, E^-) being a partition of C which is extended periodically in $\mathbb{R}^2$.

In this paper, we address *the general case* of equation (2) (not necessarily piecewise constant) and in higher dimensions as well. The aim of the present work is twofold : (i) to give a complete and rigorous mathematical treatment of these questions (the results are new even in the one dimensional case and for the patch model), and (ii) to discuss these types of problems in the framework of a general periodic environment and in higher dimensions as well. In particular, we address here from a mathematical standpoint the question of environmental fragmentation, which is an important issue in ecology. As far as we know, even in the simplified case of the periodic patch model, the results are proved rigorously here for the first time. We introduce here the method of rearrangement. As we will see in the last section, it also allows us to simplify and generalize the known results for bounded domains.

The present paper is the first one in a series of two. Here we are chiefly concerned with discussing the existence of a stationary state of (2), that is a positive solution $p(x)$ of

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) = f(x, p) \ \text{ in } \mathbb{R}^N, \\ p(x) > 0,\ x \in \mathbb{R}^N. \end{cases} \tag{5}$$

Under some assumptions which will be made precise later, solutions of (5) may turn out to be periodic. But the periodicity assumptions will not always be made a priori. Periodicity is understood here to mean L_1-periodicity in x_1,..., L_N-periodicity in x_N (assuming that $A(x)$ and $f(x, u)$ have such a dependance in x).

In problems (2) and (5), it is not easy to understand the complex interaction between more favourable and less favourable zones. Furthermore, how does the balance between diffusion and reaction play a role ? It is not obvious a priori and it may actually sometimes be counter-intuitive. We establish here a simple necessary and sufficient condition for such a solution of (5) to exist. This criterion is related to existing results in the literature. In the bounded domains case, with various boundary conditions, it is discussed in several papers of Cantrell and Cosner (see Section 6). In the unbounded domains case, with $f(x, s) = \mu(x)s - \nu(x)s^2$, the sufficient condition for existence follows from results of Pinsky [77] and Engländer, Kyprianou [34]. These authors actually consider more general operators, not necessarily in the periodic framework (see Remark 1 below[2]).

For general equation (5), we prove some new Liouville type results of independent interest. These results will be stated in the next section.

In the ecological context, existence of a solution of (5) should be viewed as a condition allowing for the survival of the species under consideration. In fact more precisely, we obtain a complete description of the asymptotic behaviour of solutions of the evolution equation (2). Under some assumptions which are made precise in section 2, we prove that when a solution

[2] We would like to thank A. E. Kyprianou for pointing out these results to us.

$p(x)$ of (5) exists, then, asymptotically, at every x, the solution $u(t,x)$ converges to that $p(x)$ as $t \to +\infty$. On the contrary, as soon as $p(x)$ ceases to exist, there is *extinction*, meaning that $u(t,x) \to 0$ as $t \to +\infty$ at every point.

Thus, within the framework of model (2), we obtain a necessary and sufficient condition for the survival of this species in a periodic environment.

Having established this criterion, we proceed to discuss various conditions under which the species survives. In particular, we treat the question of the role of fragmentation of the environment. It has been shown in [85], for the patch model in one dimension on the basis of numerical computations, that, everything else equal, having small unfavourable zones leaves less chances for survival than having one large zone (with the same total surface). This is a remarkable discovery in this model which sets on a firm theoretical ground the adverse effect of environment fragmentation. A good example to see this is to wonder whether several roads accross some forest are better or worse for species survival than one large road with the same total width. Results of the same kind have been established but in the framework of bounded intervals with boundary conditions, in a series of papers by Cantrell and Cosner, who have also further discussed systems in [29].

In this paper, we prove this result rigorously. Further, we actually derive a much more general result. For instance, a consequence of our finding is that assuming a patch model with k types of habitat (including the case $k = 2$ as before), we derive the optimal arrangement of these zones in order to allow for species survival. We establish a result in higher dimensions as well and prove that symmetric connected patches are more favourable than disconnected domains. The optimal shape, however, is not known in higher dimensions and this leads to interesting open problems. Actually, the methods which we introduce in this paper using various rearrangements also allow us to extend the results of Cantrell and Cosner in the framework of bounded domains to more general nonlinearities. Precise results are discussed in Section 6.

Further, we analyze the effects of high amplitude. One of the results we establish here is that it suffices to have a very favourable (even quite narrow) zone to allow for species survival, no matter how bad the environment may be elsewhere.

All these properties bear consequences for species survival but also shed light on conditions needed to eradicate invading biological species.

In a forthcoming paper [18], we analyze the question of invasion for problems of the type (2). More precisely, we connect the necessary and sufficient condition for species survival to that for propagation of pulsating fronts invading the uniform state 0 (see [85, 93] for the definition and [13, 16, 56, 92, 93] for some related mathematical results). We further obtain a variational formula for the minimal speed of propagation of such fronts.

The present paper is organized as follows. In the next section, we set the mathematical framework and state all the main results. In Section 3, we prove uniqueness and existence results for solutions of (5). In particular, there we establish a nonlinear Liouville theorem. Next, in Section 4, we give some stability results concerning the long time behaviour of solutions of problem (2) with initial data. In Section 5, we apply the general results to some special classes of functions f arising in some biological models, and we state some "species persistence" results. Lastly, Section 6 summarizes some facts on the bounded domains case with different types of boundary conditions.

2 Statements of the main results

We are concerned here with equation

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}^N, \tag{6}$$

and its stationary solutions given by

$$-\nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \mathbb{R}^N. \tag{7}$$

Let L_1,...,$L_N > 0$ be N given real numbers. In the following, saying that a function $g : \mathbb{R}^N \to \mathbb{R}$ is periodic means that $g(x_1, \cdots, x_k + L_k, \cdots, x_N) \equiv g(x_1, \cdots, x_N)$ for all $k = 1, \cdots, N$. Let C be the period cell defined by

$$C = (0, L_1) \times \cdots \times (0, L_N).$$

Let us now describe the precise assumptions. Throughout the paper, the diffusion matrix field $A(x) = (a_{ij}(x))_{1 \le i,j \le N}$ is assumed to be periodic, of class $C^{1,\alpha}$ (with $\alpha > 0$), and uniformly elliptic, in the sense that

$$\exists \alpha_0 > 0, \ \forall x \in \mathbb{R}^N, \ \forall \xi \in \mathbb{R}^N, \quad \sum_{1 \le i,j \le N} a_{ij}(x)\xi_i\xi_j \ge \alpha_0 |\xi|^2. \tag{8}$$

The function $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is of class $C^{0,\alpha}$ in x locally in u, locally Lipschitz-continuous with respect to u, periodic with respect to x. Moreover, assume that $f(x, 0) = 0$ for all $x \in \mathbb{R}^N$, that f is of class C^1 in $\mathbb{R}^N \times [0, \beta]$ (with $\beta > 0$), and set $f_u(x, 0) := \lim_{s \to 0^+} f(x, s)/s$. Unless otherwise specified, the assumptions above are made throughout the paper. In Remark 2 below we also explain how to include in our results the case of the patch model which involves terms $f(x, u)$ and $A(x)$ which are discontinuous with respect to x.

In several results below, the function f is furthermore assumed to satisfy

$$\forall x \in \mathbb{R}^N, \quad s \mapsto f(x,s)/s \text{ is decreasing in } s > 0 \tag{9}$$

and/or

$$\exists M \ge 0, \ \forall s \ge M, \ \forall x \in \mathbb{R}^N, \quad f(x,s) \le 0. \tag{10}$$

Examples of functions f satisfying (9-10) are functions of the type (3) or (4), namely $f(x, u) = u(\mu(x) - \nu(x)u)$ or simply $f(x, u) = u(\mu(x) - u)$, where μ and ν are periodic.

The criterion of existence (as well as uniqueness and asymptotic behaviour) is formulated with the principal eigenvalue λ_1 of the operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0\phi := -\nabla \cdot (A(x)\nabla\phi) - f_u(x, 0)\phi,$$

with periodicity conditions. Namely, we define λ_1 as the unique real number such that there exists a function $\phi > 0$ which satisfies

$$\begin{cases} -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1\phi \text{ in } \mathbb{R}^N, \\ \phi \text{ is periodic}, \ \phi > 0, \ \|\phi\|_\infty = 1. \end{cases} \tag{11}$$

Let us recall that ϕ is uniquely defined by (11).

With a slight abuse of definition, in the following we say that 0 is an *unstable solution* of (7) if $\lambda_1 < 0$. The stationary state 0 is said to be *stable* otherwise, *i.e.* if $\lambda_1 \ge 0$. These definitions will be seen to be natural in view of the results we prove here.

2.1 Existence and uniqueness results

We are now ready to state the existence and uniqueness result on problem (7). Let us start with the criterion for existence.

Theorem 1 *1) Assume that f satisfies* (10) *and that* 0 *is an unstable solution of* (7) (*that is* $\lambda_1 < 0$). *Then, there exists a* positive *and periodic solution p of* (7).

2) Assume that f satisfies (9) *and that* 0 *is a stable solution of* (7) (*that is* $\lambda_1 \geq 0$). *Then there is no positive bounded solution of* (7) (*i.e.* 0 *is the only* nonnegative *and bounded solution of* (7)).

Remark 1 For special nonlinearities f of the type $f(x,u) = u(\mu(x) - \nu(x)u)$, where μ and ν may not be periodic anymore, and for a more general *non-divergence* elliptic operator like $-\nabla \cdot (A(x)\nabla u) + B(x) \cdot \nabla u = f(x,u)$ with drift B, the above results have a probabilistic interpretation. Such equations arise in the theory of branching processes. In this framework, and for nonlinearities $f(x,u) = u(\mu(x) - \nu(x)u)$, Part 1) of Theorem 1 is due to Engländer, Pinsky [35] (see also Engländer, Kyprianou [34] and Pinsky [77] and Remark 5 below). We are much grateful to J. Engländer, A. Kyprianou and R.G. Pinsky for several useful comments about this literature and on the relationship with the probabilistic point of view.
In the bounded domains case with Dirichlet, Neumann or Robin boundary conditions, the same type of results as in Theorem 1 can be found in [70] (in dimension 1) or [27] (in higher dimensions, with constant diffusion). We refer to Section 6 for more details about the bounded domains case.

Remark 2 We have assumed that $f(x,u)$ was (at least) continuous with respect to x and u. In fact, one can easily extend these results to more general classes of f which cover, in particular, the case of the patch model. The more general statement assumes the following :

(i) $f(x,s)$ is measurable in x and bounded, uniformly on compact sets of $s \in [0, +\infty)$,

(ii) $f(x,s) \leq 0$ for all $s \geq M$, a.e. $x \in \mathbb{R}^n$,

(iii)There is some periodic bounded measurable function $\mu \in L^\infty(\mathbb{R}^n)$ such that $f(x,s) \leq \mu(x)s$ for all $s \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$,

(iv) For all $\varepsilon > 0$ there exists $\delta > 0$, such that $f(x,s) \geq \mu(x)s - \varepsilon s$ for all $s \in [0, \delta)$ and a.e. $x \in \mathbb{R}^N$

(v) assumption (9) is understood a.e. $x \in \mathbb{R}^N$.

Notice that the eigenvalue problem (11) is still well defined. When $s \to f(x,s)$ is C^1, one necessarily must take $\mu(x) = f_u(x,0) \in L^\infty(\mathbb{R}^N)$. Lastly, the asumptions are satisfied by nonlinear terms of the type $f(x,s) = \mu(x)s - \nu(x)s^2$, when $\mu, \nu \in L^\infty(\mathbb{R}^N)$.
Part 2) still holds good if, instead of (9), one only assumes that, for any $\beta > 0$, there is $\epsilon > 0$ such that $f(x,s) \leq f_u(x,0)s - \epsilon$ for all $x \in \mathbb{R}^N$ and $s \geq \beta$. Part 2) also holds good if, instead of (9), the function f is assumed to be such that $f(x,s)/s$ is nonincreasing in $s > 0$ for all $x \in \mathbb{R}^N$, and (strictly) decreasing at least for some x : we would like to thank R.G. Pinsky for pointing out this fact.
In the following, for simplicity, we write the proofs under the more stringent assumptions of the theorem, but the arguments are readily extended to handle this more general framework.

Next we state our uniqueness result.

Theorem 2 *Assume that f satisfies* (9) *and that* 0 *is an unstable solution of* (7) *(that is $\lambda_1 < 0$). Then, there exists at most one* positive *and bounded solution of* (7). *Furthermore, such a solution, if any, is periodic with respect to x. If $\lambda_1 \geq 0$ and f satisfies* (9), *then there is no nonnegative bounded solution of* (7) *other than* 0.

Remark 3 The last part of this theorem was already included in Theorem 1 above. We repeat it here for the statement to cover both cases $\lambda_1 < 0$ and $\lambda_1 \geq 0$.
This theorem is a Liouville type result for problem (7). Notice that the solutions of (7) are not *a priori* assumed to be periodic in x. The core part in Theorem 2 consists in proving that any positive solution of (7) is actually bounded from below by a positive constant (see Proposition 2 below), which does not seem to be a straightforward property.

2.2 Large time behaviour

Let us now consider the parabolic equation (6), and let $u(t,x)$ be a solution of (6), with initial condition $u(0,x) = u_0(x)$ in $\mathbb{R}^N$. The asymptotic behaviour of $u(t,x)$ as $t \to +\infty$ is described in the following theorem :

Theorem 3 *Assume that f satisfies* (9) *and* (10)*. Let u_0 be an arbitrary bounded and continuous function in $\mathbb{R}^N$ such that $u_0 \geq 0$, $u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of* (6) *with initial datum $u(0,x) = u_0(x)$.*

1) If 0 is an unstable solution of (7) *(that is $\lambda_1 < 0$), then $u(t,x) \to p(x)$ in $C^2_{loc}(\mathbb{R}^N)$ as $t \to +\infty$, where p is the unique positive solution of* (7) *given by Theorems 1, part 1), and 2.*

2) If 0 is a stable solution of (7) *(that is $\lambda_1 \geq 0$), then $u(t,x) \to 0$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$.*

Remark 4 In the above statement, the solution $u(t,x)$ is the (unique) minimal solution of (6) with initial condition u_0, in the sense that

$$u(t,x) = \lim_{m \to +\infty} u_m(t,x),$$

where u_m solves (6) with initial condition $u_{0,m}$, and the family $(u_{0,m})_{m \in \mathbb{N}}$ is any given nondecreasing sequence of nonnegative, smooth, compactly supported functions which converge to u_0 locally uniformly in $\mathbb{R}^N$. Actually, each $u_m(t,x)$ is itself the limit of $u_{m,n}(t,x)$ as $n \to +\infty$, where $u_{m,n}$ solves

$$(u_{m,n})_t - \nabla \cdot (A(x) \nabla u_{m,n}) = f(x, u_{m,n}), \quad t > 0, \ x \in B_n$$

with initial condition $u_{0,m}$ in B_n and boundary condition $u_{m,n}(t,x) = 0$ for $t > 0$ and $x \in \partial B_n$, where B_n is the open euclidean ball with center 0 and radius n (for n in $\mathbb{N}$ large so that B_n contains the support of $u_{0,m}$). For nonlinearities of the type $f(x,s) = \mu(x)s - \nu(x)s^p$ with $p > 1$, where μ and ν are periodic and $\inf_{\mathbb{R}^N} \nu > 0$, it follows from Theorem 2 of Engländer and Pinsky [36] that this minimal solution u is the unique solution of (6) with initial condition u_0. Without the periodicity assumption and with more general non-selfadjoint operators with unbounded coefficients, the subtle question of the uniqueness or nonuniqueness of the solutions of (6) with a given initial condition u_0 is discussed in [36].

Let us now consider the particular case $f(x,u) = u(\mu(x) - u)$ for $u \geq 0$, where μ is periodic with respect to x. Such nonlinearities arise in ecological models of species conservation and biological invasions (see section 1 for the motivation and [18, 56] for propagation phenomena related to these equations). Such a function $f(x,u) = u(\mu(x) - u)$ fulfills conditions (9) and (10). Following Remark 2, we may actually relax the regularity assumptions.

Gathering all the previous results, the following corollary holds :

Corollary 1 *Let $f(x,u) = u(\mu(x) - u)$ for $u \geq 0$, where μ is in $L^\infty(\mathbb{R}^N)$ and periodic. Let $u_0 \geq 0$, $u_0 \not\equiv 0$ be a bounded and uniformly continuous function in $\mathbb{R}^N$ and let $u(t,x)$ be the solution of* (6) *with initial datum $u(0,x) = u_0(x)$.*

1) If 0 is an unstable solution of (7) *(that is $\lambda_1 < 0$) then there exists a unique bounded* positive *solution p of* (7)*, and $u(t,x) \to p(x)$ locally in x as $t \to +\infty$.*

2) If 0 is a stable solution of (7) *(that is $\lambda_1 \geq 0$), then 0 is the unique* nonnegative *bounded solution of* (7)*, and $u(t,x) \to 0$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$.*

2.3 Effects of the heterogeneity on species survival

Let us denote by $\lambda_1[\mu]$ the first eigenvalue of (11) with $f_u(x,0) = \mu(x)$. From the previous results, we see that, in this model, the survival of the species or its extinction hinge on the sign of $\lambda_1[\mu]$. Furthermore, we show in [18] that this sign also determines biological invasions in the form of travelling front-like solutions (actually *pulsating travelling fronts*). Hence it is of particular interest to investigate how the various factors such as the shape of $\mu(x)$, the distribution of unfavourable zones or large amplitude oscillations in $\mu(x)$, affect the sign of $\lambda_1[\mu]$. The next series of results discuss these effects.

2.3.1 Distribution effects

Let us first discuss the influence of a heterogeneous function μ, that is that μ depends on x, as compared to the case where μ would be constant with the same average.

Proposition 1 *Under the above assumptions, one has*

$$\lambda_1[\mu] \leq \lambda_1[\mu_0] = -\mu_0,$$

where $\mu_0 = \dfrac{1}{|C|}\displaystyle\int_C \mu$ and $|C|$ denotes the Lebesgue measure of the cell of periodicity $C = (0,L_1) \times \cdots \times (0,L_N)$.

Let us now study the influence of the repartition of μ, assuming that the distribution function of μ is given.

Let us first discuss the one-dimensional periodic patch model described in the introduction. There, we assume that

$$\mu(x) = \begin{cases} \mu^+ & \text{in } E_+ \subset \mathbb{R}, \\ \mu^- & \text{in } E_- = \mathbb{R} \backslash E_+. \end{cases}$$

Consider now another function $\mu^*(x)$ having the same distribution function as μ but where the unfavourable zone is an interval in any periodicity cell. That is, we set

$$\mu^*(x) = \begin{cases} \mu^+ & \text{in } E_+^* \subset \mathbb{R}, \\ \mu^- & \text{in } E_-^* = \mathbb{R} \backslash E_+^*, \end{cases}$$

with μ^* being L-periodic, as is μ, and $E_-^* \cap (0,L)$ is a (connected) interval. The question we want to solve is to know which of the two configurations leaves most chances for survival.

Following the metaphor of Shigesada and Kawasaki [85], one can think of a forest in which a periodic array of parallel roads are cut through. The forest is thought of as a favourable homogeneous medium and roads as an unfavourable homogeneous medium with a constant negative reproduction rate $\mu_- < 0$ (or death rate $|\mu_-|$). The question here is to know whether several small forest roads, say of widths l_1,...,l_p, in a given periodicity cell, are better –in the sense of species survival– than one big road of width $l_1 + \cdots + l_p$. Relying on numerical calculations, Shigesada and Kawasaki [85] have observed that the latter leaves more chances for species survival (see also Section 6 for a discussion on such results with other boundary conditions). This remarkable finding illustrates *the adverse effect of environment fragmentation on species survival* which is important in ecology. Incidentally, this is also relevant to eradication techniques of certain harmful species.

Here, we actually prove this result rigorously.

Theorem 4 *With the above notations, assuming that the diffusion coefficient $A(x)$ is a constant positive real number, one has*

$$\lambda_1[\mu^*] \leq \lambda_1[\mu].$$

Therefore, whenever μ allows for survival, so does μ^* but in some cases, μ^* will allow for survival while μ will not. It is indeed simple to construct examples where $\lambda_1[\mu^*] < 0 < \lambda_1[\mu]$.

We actually prove a much more general result, in arbitrary dimension, and for general reaction terms $f(x,u)$. The previous proposition is a particular case of it.

To state our result, we need to introduce the notion of Schwarz and Steiner periodic symmetrizations of a function. For more details and properties about these notions, we refer the reader to the monograph of B. Kawohl [65].

Consider a L-periodic function $\mu(x)$ defined on the real line $\mathbb{R}$. There exists a unique function $\mu^*(x)$, L-periodic on $\mathbb{R}$, satisfying the following properties :

(i) μ^* is symmetric with respect to $x = L/2$ and μ^* is nondecreasing on $[0,L]$ away from the symmetry center $L/2$, i.e.

$$\text{for all } x, y \in [0,L], \quad \mu^*(x) \leq \mu^*(y) \text{ if } |x - L/2| \leq |y - L/2|$$

(ii) μ^* has the same distribution function as μ, that is :

$$\text{meas } \{x \in (0,L);\ \mu(x) \leq \alpha\} = \text{meas } \{x \in (0,L);\ \mu^*(x) \leq \alpha\}$$

for all real α.

This function μ^* is called the Schwarz periodic rearrangement. An example of it is given in Figure 1 below.

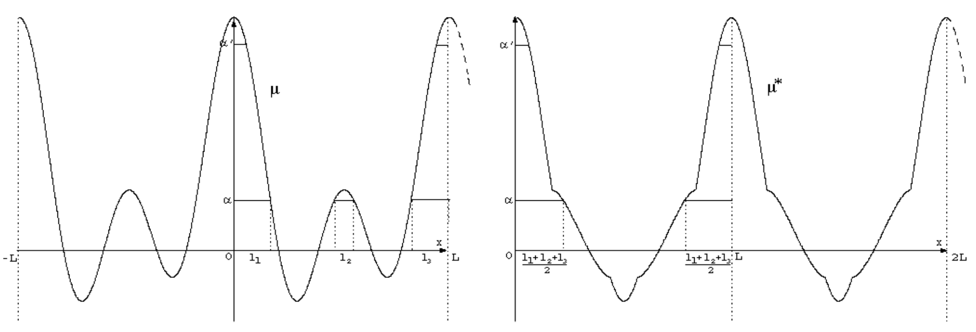

FIG. 1 – A function μ and its periodic Schwarz rearrangement μ^*

Consider now a function $\mu(x)$ periodic in $\mathbb{R}^N$ with a period cell $C = (0, L_1) \times \cdots \times (0, L_N)$. Keeping fixed all other variables but x_k, we can rearrange as above the function $\mu(x)$ with respect to x_k. This is called Steiner periodic rearrangement in the variable x_k. By performing such Steiner periodic rearrangements successively on all variables $x_1, x_2, \ldots, x_N$, we obtain a new function, $\mu^*(x_1, \cdots, x_N)$. Thus, this function is periodic in all variables, symmetric and decreasing.

Theorem 5 *Assume that the diffusion matrix A is the identity matrix and denote by $\lambda_1[f_u(x, 0)]$ the principal eigenvalue of* (11) *involving $f_u(x, 0)$. Let $f_u^*(\cdot, 0)$ be the successive Steiner symmetrizations of $f_u(\cdot, 0)$ in the variables $x_1, \ldots, x_N$.*

Then,

$$\lambda_1[f_u^*(\cdot, 0)] \le \lambda_1[f_u(\cdot, 0)].$$

As already pointed out, this covers the case of the patch model. Even in this case, in higher dimension, say $N = 2$, this property is new. An example of how unfavourable zones are assembled by Steiner rearrangement is described in Figure 2 below.

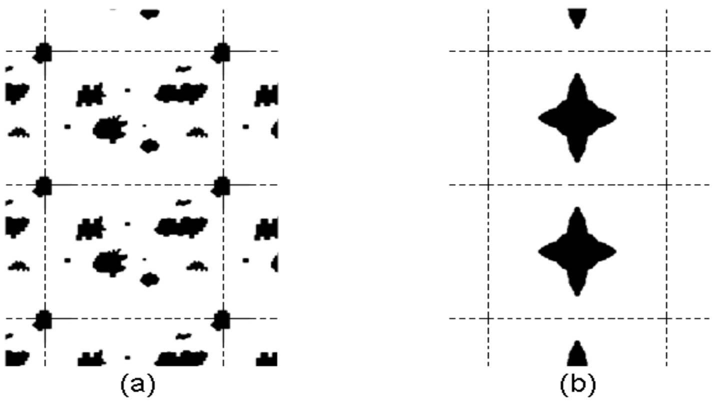

FIG. 2 – The effect of Steiner symmetrizaton on unfavourable zones. Distribution of unfavorable zones : (a) for $\mu(x)$ and (b) for rearranged $\mu^*(x)$ successively in the variables x_1 and x_2.

Theorem 5 shows that the μ^* configuration leaves better survival chances. Hence, the more the unfavourable zones are concentrated, the better the chances of survival of the species. Note that the result of the succession of Steiner symmetrizations will depend on the order in which the variables are taken. This result supports the adverse effect of fragmentation of the environment on species persistence. It holds not only in the periodic patch model when $\mu(x)$ takes two values, but for an arbitrary function $\mu(x)$ (also one taking several values). Note, however, that, in the patch model, for a given total area of unfavourable environment in one periodicity cell, the optimal shape (i.e. the one that minimizes $\lambda_1[\mu]$) is not known. It could actually be like in figure 3 below. We pose as an open problem, to determine the optimal shape and to derive its properties.

2.3.2 Effects of the amplitude of the heterogeneity

The following result is concerned with the study of the influence of the size of the nonlinearity f. To stress this effect, we now call $\lambda_1(f)$ the first eigenvalue of (11) with the nonlinearity f.

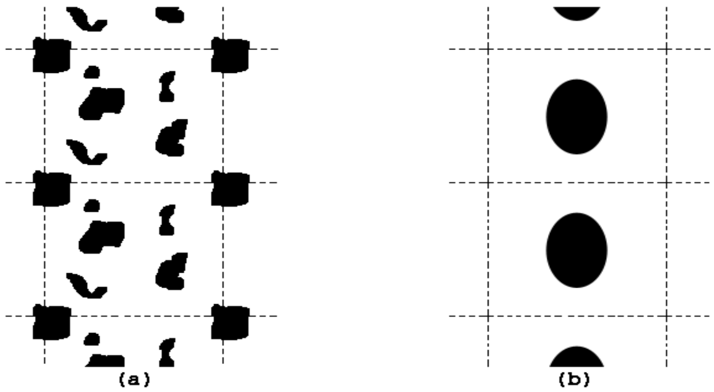

FIG. 3 – An example of a rearrangement which is Steiner symmetric in both variables but is not obtained by the procedure described here.

Consider the problem

$$-\nabla \cdot (A(x)\nabla u) = Bf(x,u) \text{ in } \mathbb{R}^N, \tag{12}$$

where $B > 0$ is a given positive real number and f satisfies assumptions (9) and (10). As it follows from Theorem 1 and 2, this problem admits a positive periodic solution if and only if 0 is an unstable solution of (12). Let us examine the effect of the amplitude factor B. The following theorem below holds for general functions f :

Theorem 6 *1) If* $\int_C f_u(x,0) > 0$*, or if* $\int_C f_u(x,0) = 0$ *and* $f_u(x,0) \not\equiv 0$*, then* $\lambda_1(Bf) < 0$ *for every* $B > 0$*, and the function* $B \mapsto \lambda_1(Bf)$ *is decreasing in* $B \geq 0$*.*

2) If $\int_C f_u(x,0) < 0$*, then* $\lambda_1(Bf) > 0$ *for all* $B > 0$ *small enough. Assume that there exists* $x_0 \in C$ *such that* $f_u(x_0,0) > 0$*. Then* $\lambda_1(Bf) < 0$ *for* B *large enough, and the function* $B \mapsto \lambda_1(Bf)$ *is decreasing in* B *for* B *large enough.*

Likewise, if we assume that f has a dependence with respect to one parameter B –we write $f = f^B$– such that $f_u^B(x,0) = h(x) + Bg(x)$ for some $h, g \in L^\infty$ and assuming that $g > 0$ on some set of positive measure, we can prove the following. For large B (no matter how h and g are distributed), there is always survival. The proofs are the same as for Theorem 6 and will not be detailed separately.

As a consequence of the last theorem, we can say that increasing the size of the nonlinearity, assuming that the favourable region is not empty, enhances the chance of having 0 unstable. Hence, it increases the chance of biological survival (existence of a positive solution p of (7)). In our forthcoming paper [18], we show that it also increases the speed of biological invasion. This result bears consequences on species survival, as well as on techniques of eradication for harmful species.

3 Existence and uniqueness of a stationary solution

We start with existence which is a simpler aspect here.

3.1 Proof of existence

Assume first that 0 is an unstable solution of (7) and that condition (10) is fulfilled. Let us prove that there exists a positive and periodic solution of (7). Let ϕ be the unique positive solution of

$$\begin{cases} -\nabla\cdot(A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1\phi \ \text{ in } \mathbb{R}^N, \\ \phi \text{ is periodic}, \ \phi > 0, \ \|\phi\|_\infty = 1, \end{cases} \tag{13}$$

with $\lambda_1 < 0$. Since $f(x,u)$ is of class C^1 in $\mathbb{R}^N \times [0,\beta]$ (with $\beta > 0$), for $\kappa > 0$ small enough, one gets :

$$f(x,\kappa\phi) \geq \kappa\phi f_u(x,0) + \frac{\lambda_1}{2}\kappa\phi \ \text{ in } \mathbb{R}^N. \tag{14}$$

Therefore, it follows that

$$-\kappa\nabla\cdot(A(x)\nabla\phi) - f(x,\kappa\phi) \leq \frac{\lambda_1}{2}\kappa\phi \leq 0 \ \text{ in } \mathbb{R}^N, \tag{15}$$

and $\kappa\phi$ is a subsolution of (7) with periodicity conditions. Moreover, if M is taken as in (10), the constant M is an upper solution of (7) with periodicity conditions, and (for κ small enough) $\kappa\phi \leq M$ in $\mathbb{R}^N$. Thus, it follows from a classical iteration method that there exists a periodic classical solution p of (7) which satisfies $\kappa\phi \leq p \leq M$ in $\mathbb{R}^N$. Theorem 1, part 1) is proved.

Next, assume that p is a nonnegative bounded solution of (7) and assume that 0 is stable ($\lambda_1 \geq 0$). Let ϕ be the first eigenfunction of (13). From hypothesis (9), one has $f(x,\gamma\phi(x)) < f_u(x,0)\gamma\phi(x)$ for all $x \in \mathbb{R}^N$ and $\gamma > 0$. Hence,

$$-\nabla\cdot(A(x)\nabla(\gamma\phi)) - f(x,\gamma\phi) > \lambda_1\gamma\phi \geq 0 \text{ in } \mathbb{R}^N \tag{16}$$

for all $\gamma > 0$.

Recall that p is a nonnegative and bounded solution of (7). Since ϕ is bounded from below away from 0 and p is bounded, one can define

$$\gamma^* = \inf\left\{\gamma > 0, \ \gamma\phi > p \text{ in } \mathbb{R}^N\right\} \geq 0. \tag{17}$$

Assume that $\gamma^* > 0$, and set $z := \gamma^*\phi - p$. Then $z \geq 0$, and there exists a sequence $x_n \in \mathbb{R}^N$ such that $z(x_n) \to 0$ as $n \to +\infty$.

Assume at first that up to the extraction of some subsequence, $x_n \to \overline{x} \in \mathbb{R}^N$ as $n \to +\infty$. By continuity, $z(\overline{x}) = 0$. As $\gamma^*\phi$ is a supersolution of (7) (in the sense that it satisfies (16)) with periodicity conditions, it is easy to see from the strong elliptic maximum principle that $z \equiv 0$. Therefore $p \equiv \gamma^*\phi$ is a positive and periodic solution of (7). Since $\lambda_1 \geq 0$ (0 is assumed to be stable), it follows from (7) and (16) that

$$0 = -\nabla\cdot(A(x)\nabla p) - f(x,p(x)) > 0.$$

One is thus led to a contradiction.

In the general case, let $(\overline{x}_n) \in \overline{C}$ be such that $x_n - \overline{x}_n \in \prod_{i=1}^N L_i\mathbb{Z}$. Then, up to the extraction of some subsequence, one can assume that there exists $\overline{x}_\infty \in \overline{C}$ such that $\overline{x}_n \to \overline{x}_\infty$ as $n \to +\infty$. Next, set $\phi_n(x) = \phi(x+x_n)$ and $p_n(x) = p(x+x_n)$. Since both A and f are periodic with respect to x, the functions $\gamma^*\phi_n$ and p_n satisfy

$$\begin{array}{r} -\nabla\cdot(A(x+\overline{x}_n)\nabla(\gamma^*\phi_n)) - f(x+\overline{x}_n,\gamma^*\phi_n) > 0 \\ -\nabla\cdot(A(x+\overline{x}_n)\nabla p_n) - f(x+\overline{x}_n,p_n) = 0 \end{array} \ \text{ in } \mathbb{R}^N. \tag{18}$$

From standard elliptic estimates, it follows that (up to the extraction of some subsequences) p_n converge in C^2_{loc} to a function p_∞ satisfying

$$-\nabla\cdot(A(x+\overline{x}_\infty)\nabla p_\infty) - f(x+\overline{x}_\infty, p_\infty) = 0 \text{ in } \mathbb{R}^N, \tag{19}$$

while the sequence $(\gamma^*\phi_n)$ converges to $\gamma^*\phi_\infty := \gamma^*\phi(\cdot+\overline{x}_\infty)$, and

$$-\nabla\cdot(A(x+\overline{x}_\infty)\nabla(\gamma^*\phi_\infty)) - f(x+\overline{x}_\infty, \gamma^*\phi_\infty) > 0 \text{ in } \mathbb{R}^N. \tag{20}$$

Let us set $z_\infty(x) := \gamma^*\phi_\infty(x) - p_\infty(x)$. Then

$$z_\infty(x) = \lim_{n\to+\infty}\left[\gamma^*\phi(x+x_n) - p(x+x_n)\right],$$

whence $z_\infty(x) = \lim_{n\to+\infty} z(x+x_n)$. Therefore $z_\infty \geq 0$ and $z_\infty(0) = 0$. It then follows from the strong maximum principle that $z_\infty = 0$ and reaches a contradiction as above.

Finally, in all the cases, one has $\gamma^* = 0$, thus $p \equiv 0$, and the proof of Theorem 1 is complete. □

Remark 5 Theorem 1 holds, as such, if equation (7) is replaced by

$$-\nabla\cdot(A(x)\nabla u) + B(x)\cdot\nabla u = f(x,u) \ \text{ in } \mathbb{R}^N, \tag{21}$$

where B is a $C^{0,\alpha}$ periodic drift. Indeed, the proof of Theorem 1 does not rely on the variational structure of (7).
More generally, consider the case where A, B and f are not periodic (with respect to x) anymore. One can wonder whether a result similar to Theorem 1 still holds in this case. For this purpose, a possible generalization of the first eigenvalue λ_1 of the operator $\mathcal{L} = -\nabla\cdot(A(x)\nabla) + B(x)\cdot\nabla - f_u(x,0)$ in $\mathbb{R}^N$ is

$$\begin{aligned} \lambda_1 &= \inf\left\{\lambda\in\mathbb{R},\ \exists\varphi\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\ \varphi>0,\ (\mathcal{L}-\lambda)\varphi\leq 0 \ \text{ in } \mathbb{R}^N\right\} \\ &= \inf_{\varphi\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\,\varphi>0}\ \sup_{x\in\mathbb{R}^N}\left(\frac{\mathcal{L}\varphi(x)}{\varphi(x)}\right). \end{aligned}$$

This definition for λ_1 gives the same value as before in the periodic case. With the same arguments as above, one can easily prove that if f satisfies (10) and if 0 is an unstable solution of (21) ($\lambda_1 < 0$), then there exists a positive solution p of (21). On the other hand, if f satisfies (9) and if 0 is a strictly stable solution of (21) ($\lambda_1 > 0$), then there is no positive bounded solution of (21) (i.e. 0 is the only nonnegative and bounded solution of (21)). For the proof, assume indeed that there is a positive bounded solution p of (21) ; it follows from (9) that $\mathcal{L}p \leq 0$, whence $\lambda_1 \leq 0$. However, it is not clear whether, under assumption (9), the nonexistence of positive bounded solutions p of (21) still holds if $\lambda_1 = 0$. We mention this question as an open problem.
Another possible generalized first eigenvalue of $\mathcal{L}$ in the non-periodic case is the following

$$\begin{aligned} \lambda_1' &= \sup\left\{\lambda\in\mathbb{R},\ \exists\varphi\in C^2(\mathbb{R}^N),\ \varphi>0,\ (\mathcal{L}-\lambda)\varphi\geq 0 \ \text{ in } \mathbb{R}^N\right\} \\ &= \sup_{\varphi\in C^2(\mathbb{R}^N),\,\varphi>0}\ \inf_{x\in\mathbb{R}^N}\left(\frac{\mathcal{L}\varphi(x)}{\varphi(x)}\right). \end{aligned}$$

In the case of a bounded smooth domain, this definition reduces to the classical first eigenvalue of $\mathcal{L}$ with Dirichlet boundary conditions (see [25], [75]). In the periodic case in $\mathbb{R}^N$, one has $\lambda_1 \leq \lambda_1'$, but with a strict inequality in general, even in the case of constant coefficients (for instance, for

$\mathcal{L}u = -u'' + u'$ in $\mathbb{R}$, one has $0 = \lambda_1 < \lambda'_1 = 1/4$), see [2], [19], [76]. But this definition of λ'_1 is well-suited for a condition on the existence of other types of solutions of (21), maybe not bounded, in the general nonperiodic case. Namely, for a function f of the type $f(x,s) = \mu(x)s - \nu(x)s^2$, Pinsky [77] (see also [34]) proved that the existence of a solution of minimal growth at infinity for (21) is equivalent to $\lambda'_1 < 0$ (a solution of minimal growth at infinity for (21) is a positive solution u of (21) such that $u \le v$ in $\mathbb{R}^N \backslash D$ for all bounded domain D and for all nonnegative solution v of (21) in $\mathbb{R}^N \backslash D$ with $u \le v$ on ∂D).

3.2 Proof of uniqueness

For analogous problems on bounded domains with e.g. Dirichlet conditions on the boundary, uniqueness of the positive solutions is well known (compare [10]). The difficulty here arises because of the lack of compactness and because of the fact that one does not assume a priori that u is bounded from below away from zero.

The proof of Theorem 2 essentially relies on the following property.

Proposition 2 *Assume that* 0 *is an unstable solution of* (7)*. Let $u \in C^2(\mathbb{R}^N)$ be a bounded nonnegative solution of* (7)*. Then, either $u \equiv 0$ or there exists $\varepsilon > 0$ such that $u(x) \ge \varepsilon$ for all $x \in \mathbb{R}^N$.*

Note that periodic solutions obviously satisfy this property. But, here, we look at uniqueness within a more general class of functions. In particular, it is not assumed *a priori* that $\inf_{\mathbb{R}^N} u > 0$, which we will now rule out.

We prove Proposition 2 through a succession of lemmas. Let B_R be the open ball of $\mathbb{R}^N$, with centre 0 and radius R. Let y be an arbitrary point in $\mathbb{R}^N$. It is well-known that there exist a unique real number (principal eigenvalue) λ_R^y, and a unique function φ_R^y (principal eigenfunction) in $C^2\left(\overline{B_R}\right)$, satisfying[3]

$$\left\{ \begin{array}{rcll} -\nabla \cdot (A(x+y)\nabla \varphi_R^y) - f_u(x+y,0)\varphi_R^y & = & \lambda_R^y \varphi_R^y & \text{in } B_R, \\ \varphi_R^y & > & 0 & \text{in } B_R, \\ \varphi_R^y & = & 0 & \text{on } \partial B_R, \\ \|\varphi_R^y\|_\infty & = & 1. & \end{array} \right. \tag{22}$$

Since both λ_R^y and φ_R^y are unique, standard elliptic estimates and compactness arguments imply that the maps $y \mapsto \varphi_R^y$ and $y \mapsto \lambda_R^y$ are continuous with respect to y (the continuity of φ_R^y is understood in the sense of the uniform topology in $\overline{B_R}$). Note that, since f is periodic in x, φ_R^y and λ_R^y are periodic with respect to y as well.

Let $\tilde{\lambda}_1^y$ be the principal eigenvalue and ϕ^y the principal eigenfunction of

$$\left\{ \begin{array}{l} -\nabla \cdot (A(x+y)\nabla \phi^y) - f_u(x+y,0)\phi^y = \tilde{\lambda}_1^y \phi^y \text{ in } \mathbb{R}^N, \\ \phi^y \text{ is periodic and positive in } \mathbb{R}^N, \\ \|\phi^y\|_\infty = 1. \end{array} \right. \tag{23}$$

First, it is straightforward to observe :

Lemma 1 *The first eigenvalue $\tilde{\lambda}_1^y$ does not depend on y. In other words, $\tilde{\lambda}_1^y = \lambda_1$ for all $y \in \mathbb{R}^N$, where λ_1 is the first eigenvalue of* (11)*.*

[3] throughout the paper, the operator ∇ always refers to the derivation with respect to the x variables

Proof. Set $\phi(x) := \phi^y(x-y)$. The function ϕ satisfies

$$\begin{cases} -\nabla\cdot(A(x)\nabla\phi) - f_u(x,0)\phi = \tilde{\lambda}_1^y\phi \text{ in } \mathbb{R}^N, \\ \phi \text{ is periodic and positive in } \mathbb{R}^N, \\ \|\phi\|_\infty = 1. \end{cases} \tag{24}$$

Therefore, by uniqueness, one has $\phi = \phi^0$, and $\tilde{\lambda}_1^y = \tilde{\lambda}_1^0 = \lambda_1$. □

Lemma 2 *For all $y \in \mathbb{R}^N$ and $R > 0$, one has $\lambda_R^y > \lambda_1$.*

Proof. The function φ_R^y satisfies

$$\begin{cases} -\nabla\cdot(A(x+y)\nabla\varphi_R^y) - f_u(x+y,0)\varphi_R^y & & & \\ \qquad -\lambda_1\varphi_R^y & = & (\lambda_R^y-\lambda_1)\varphi_R^y & \text{in } B_R, \\ \varphi_R^y & > & 0 & \text{in } B_R, \\ \varphi_R^y & = & 0 & \text{on } \partial B_R, \\ \|\varphi_R^y\|_\infty & = & 1. & \end{cases} \tag{25}$$

Assume that $\lambda_R^y \le \lambda_1$. Let ϕ^y be the solution of (23). Then ϕ^y satisfies

$$\begin{cases} -\nabla\cdot(A(x+y)\nabla\phi^y) - f_u(x+y,0)\phi^y & = & \lambda_1\phi^y & \text{in } B_R, \\ \phi^y & > & 0 & \text{in } \overline{B_R}. \end{cases} \tag{26}$$

Since $\phi^y > 0$ in $\overline{B_R}$, one can assume that $\kappa\varphi_R^y < \phi^y$ in $\overline{B_R}$ for all $\kappa > 0$ small enough. Now, set

$$\kappa^* := \sup\left\{\kappa > 0,\ \kappa\varphi_R^y < \phi^y \text{ in } \overline{B_R}\right\} > 0.$$

Then, by continuity, $\kappa^*\varphi_R^y \le \phi^y$ in $\overline{B_R}$ and there exists x_1 in $\overline{B_R}$ such that $\kappa^*\varphi_R^y(x_1) = \phi^y(x_1)$. But, since $\phi^y > 0$ in $\overline{B_R}$ and $\varphi_R^y = 0$ on ∂B_R, it follows that $x_1 \in B_R$.

On the other hand, the assumption $\lambda_R^y \le \lambda_1$ implies, from (25), that

$$-\nabla\cdot(A(x+y)\nabla(\kappa^*\varphi_R^y)) - f_u(x+y,0)\kappa^*\varphi_R^y - \lambda_1\kappa^*\varphi_R^y \le 0 \ \text{ in } B_R.$$

Therefore, it follows from the strong elliptic maximum principle that $\kappa^*\varphi_R^y \equiv \phi^y$ in $\overline{B_R}$, which is impossible because of the boundary conditions on ∂B_R.

Finally, one concludes that $\lambda_R^y > \lambda_1$ (This can also be derived from a characterization in [25]). □

Lemma 3 *For all $y \in \mathbb{R}^N$, the function $R \mapsto \lambda_R^y$ is decreasing in $R > 0$.*

Proof. Let R_1 and R_2 be two positive real numbers with $R_1 < R_2$. The proof of this lemma is similar to that of Lemma 2, replacing λ_R^y by $\lambda_{R_1}^y$ and λ_1 by $\lambda_{R_2}^y$, and using the fact that $\varphi_{R_2}^y > 0$ in $\overline{B_{R_1}}$. □

The next lemma is a standard result (*see e.g.* [30]), but we include its proof here for the sake of completeness.

Lemma 4 *One has $\lim_{R\to+\infty}\lambda_R^y = \lambda_1$ uniformly in $y \in \mathbb{R}^N$.*

Proof. For $y \in \mathbb{R}^N$, call $\mathcal{L}^y$ the elliptic operator defined by $\mathcal{L}^y u := -\nabla \cdot (A(x+y)\nabla u) - f_u(x+y,0)u$. Since it is a self-adjoint operator, one has the following variational formula for λ_R^y :

$$\lambda_R^y = \min_{\psi \in H_0^1(B_R),\ \psi \not\equiv 0} Q_R^y(\psi), \tag{27}$$

where

$$Q_R^y(\psi) = \frac{\displaystyle\int_{B_R} \left[\nabla\psi \cdot (A(x+y)\nabla\psi) - f_u(x+y,0)\psi^2\right] dx}{\displaystyle\int_{B_R} \psi^2}. \tag{28}$$

Choose a family of functions $(\chi_R)_{R\geq 2}$, bounded in $C^2\left(\mathbb{R}^N\right)$ (for the usual norm) independently of R, and such that

$$\begin{cases} \chi_R(x) = 1 \text{ if } |x| \leq R-1, \\ \chi_R(x) = 0 \text{ if } |x| \geq R, \\ 0 \leq \chi_R \leq 1. \end{cases} \tag{29}$$

Set $\psi_R = \phi^y \chi_R$ where ϕ^y is the solution of (23). Then $\psi_R \in H_0^1(B_R)$ and

$$Q_R^y(\psi_R) = \frac{\displaystyle\int_{B_R} \left[\nabla\psi_R \cdot (A(x+y)\nabla\psi_R) - f_u(x+y,0)\psi_R^2\right] dx}{\displaystyle\int_{B_R} \psi_R^2}. \tag{30}$$

Integrating the numerator by parts over B_R, and using the boundary conditions on ∂B_R, one gets

$$Q_R^y(\psi_R) = \frac{\displaystyle\int_{B_R} \left[-\nabla \cdot (A(x+y)\nabla\psi_R)\ \psi_R - f_u(x+y,0)\psi_R^2\right] dx}{\displaystyle\int_{B_R} \psi_R^2}, \tag{31}$$

and, by definition of ψ_R,

$$\begin{aligned} &\int_{B_R} [-\nabla \cdot (A(x+y)\nabla\psi_R)\ \psi_R - f_u(x+y,0)\psi_R^2]dx \\ &= \int_{B_{R-1}} [-\nabla \cdot (A(x+y)\nabla\phi^y)\ \phi^y - f_u(x+y,0)\,(\phi^y)^2]dx \\ &+ \int_{B_R \setminus B_{R-1}} [-\nabla \cdot (A(x+y)\nabla(\phi^y\chi_R))\ \phi^y\chi_R - f_u(x+y,0)(\phi^y\chi_R)^2]dx. \end{aligned} \tag{32}$$

From equation (23) satisfied by ϕ^y and using that ϕ^y and χ_R are bounded in $C^2\left(\mathbb{R}^N\right)$, uniformly with respect to y and R, it follows that there exists $C \geq 0$ such that

$$\left| \int_{B_R} \left[-\nabla \cdot (A(x+y)\nabla\psi_R)\ \psi_R - f_u(x+y,0)\psi_R^2\right] dx - \lambda_1 \int_{B_{R-1}} (\phi^y)^2 \right| \leq CR^{N-1} \tag{33}$$

for all $R \geq 2$ and $y \in \mathbb{R}^N$. Likewise, one has

$$\left| \int_{B_R} \psi_R^2 - \int_{B_{R-1}} (\phi^y)^2 \right| \leq C' R^{N-1} \tag{34}$$

for some $C' \geq 0$, for all $R \geq 2$ and $y \in \mathbb{R}^N$.

But, since each function ϕ^y is continuous, positive and periodic, and since the functions ϕ^y depend continuously and periodically on y (in the sense of the uniform topology in $\mathbb{R}^N$), there exists $\alpha > 0$ such that $\phi^y(x) \geq \alpha$ for all $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$. Thus $\displaystyle\int_{B_{R-1}} (\phi^y)^2 \geq \alpha^2 |B_{R-1}|$. Therefore,

$$\frac{\displaystyle\int_{B_R} \psi_R^2}{\displaystyle\int_{B_{R-1}} (\phi^y)^2} \to 1 \text{ as } R \to +\infty, \tag{35}$$

uniformly with respect to $y \in \mathbb{R}^N$. Using (31), (33) and (34), one gets that $Q_R^y(\psi_R) \to \lambda_1$ as $R \to +\infty$, uniformly in $y \in \mathbb{R}^N$.

Next, (27) and Lemma 2 yield $\lambda_1 < \lambda_R^y \leq Q_R^y(\psi_R)$. As a consequence, $\lambda_R^y \to \lambda_1$ as $R \to +\infty$, uniformly in $y \in \mathbb{R}^N$. This completes the proof of Lemma 4. □

We are now able to complete the

Proof of Proposition 2. Let $u \in C^2(\mathbb{R}^N)$ be a nonnegative and bounded solution of (7). Let us assume that $u \not\equiv 0$. The strong maximum principle then implies that $u > 0$ in $\mathbb{R}^N$.

Since $f(x,u)$ is of class C^1 in $\mathbb{R}^N \times [0,\beta]$ (with $\beta > 0$), since $f(x,0) \equiv 0$ in $\mathbb{R}^N$ and since f is periodic with respect to x, one can choose $\kappa_0 > 0$ small enough such that

$$f(x+y, \kappa\varphi_R^y) \geq \kappa\varphi_R^y f_u(x+y,0) + \frac{\lambda_1}{2}\kappa\varphi_R^y \ \text{ in } B_R, \tag{36}$$

for all $0 < \kappa \leq \kappa_0$, $y \in \mathbb{R}^N$ and $R > 0$ (recall that $\lambda_1 < 0$, and $\varphi_R^y > 0$ in B_R).

From Lemmas 3 and 4, there exists $R_0 > 0$ such that

$$\forall\, R \geq R_0,\ \forall\, y \in \mathbb{R}^N,\ \lambda_R^y < \frac{\lambda_1}{2} < 0. \tag{37}$$

In the sequel, fix some $R \geq R_0$. Set $u^y(x) := u(x+y)$. The function u^y satisfies

$$-\nabla\cdot(A(x+y)\nabla u^y) - f(x+y, u^y) = 0 \ \text{ in } \mathbb{R}^N. \tag{38}$$

Furthermore, $\kappa_0\varphi_R^y$ satisfies

$$-\kappa_0\nabla\cdot(A(x+y)\nabla\varphi_R^y) = f_u(x+y,0)\kappa_0\varphi_R^y + \lambda_R^y\kappa_0\varphi_R^y \ \text{ in } B_R. \tag{39}$$

Thus, using (36) and (37), one has

$$-\kappa_0\nabla\cdot(A(x+y)\nabla\varphi_R^y) - f(x+y,\kappa_0\varphi_R^y) \leq (\lambda_R^y - \frac{\lambda_1}{2})\kappa_0\varphi_R^y \leq 0 \ \text{ in } B_R. \tag{40}$$

In other words, $\kappa_0\varphi_R^y$ is a sub-solution of (38).

Let us now show that $u^y > \kappa_0\varphi_R^y$ in $\overline{B_R}$. If not, there exists $0 < \kappa^* \leq \kappa_0$ and $x_1 \in \overline{B_R}$ such that $\kappa^*\varphi_R^y(x_1) = u^y(x_1)$ and $u^y \geq \kappa^*\varphi_R^y$ in $\overline{B_R}$ (remember that $u^y > 0$ in $\mathbb{R}^N$, whence $\min_{\overline{B_R}} u^y > 0$). Next, since $\varphi_R^y \equiv 0$ on ∂B_R, it follows that $x_1 \in B_R$. On the other hand, the computations above show that the function $\kappa^*\varphi_R^y$ is still a sub-solution of (38). The strong maximum principle gives that $\kappa^*\varphi_R^y \equiv u^y$ in $\overline{B_R}$, which is in contradiction with the conditions on ∂B_R.

Finally, one has $u^y > \kappa_0\varphi_R^y$ in $\overline{B_R}$, thus $u^y(0) > \kappa_0\varphi_R^y(0)$. In other words, $u(y) > \kappa_0\varphi_R^y(0)$ for all $y \in \mathbb{R}^N$. Since the function $y \mapsto \kappa_0\varphi_R^y(0)$ is periodic, continuous and positive over $\mathbb{R}^N$, there exists $\varepsilon > 0$ such that $\kappa_0\varphi_R^y(0) > \varepsilon$ for all $y \in \mathbb{R}^N$, and this completes the proof of Proposition 2. □

Let us now turn to the

Proof of Theorem 2. Let u and $p \in C^2\left(\mathbb{R}^N\right)$ be two *positive* and bounded solutions of (7). By Proposition 2, there exists $\varepsilon > 0$ such that $u \geq \varepsilon$ and $p \geq \varepsilon$ in $\mathbb{R}^N$.

Therefore, we can define the positive real number

$$\gamma^* = \sup\left\{\gamma > 0,\ u > \gamma p \text{ in } \mathbb{R}^N\right\} > 0. \tag{41}$$

Assume that $\gamma^* < 1$, and let us set $z := u - \gamma^* p \geq 0$. From the definition of γ^*, it follows that there exists a sequence $x_n \in \mathbb{R}^N$ such that $z(x_n) \to 0$ as $n \to +\infty$.

Assume first that, up to the extraction of some subsequence, $x_n \to \overline{x} \in \mathbb{R}^N$ as $n \to +\infty$. By continuity, one has $z \geq 0$ in $\mathbb{R}^N$, and $z(\overline{x}) = 0$. Moreover, z satisfies the equation

$$-\nabla\cdot(A(x)\nabla z) - f(x,u) + \gamma^* f(x,p) = 0 \text{ in } \mathbb{R}^N. \tag{42}$$

Furthermore, by assumption (9), $f(.,s)/s$ is decreasing in $\mathbb{R}_+$ and since we have assume $\gamma^* < 1$, one has $\gamma^* f(x,p) < f(x,\gamma^* p)$. Hence, (42) gives

$$-\nabla\cdot(A(x)\nabla z) - f(x,u) + f(x,\gamma^* p) > 0 \text{ in } \mathbb{R}^N. \tag{43}$$

Since f is locally Lipschitz continuous in the second variable, one infers from (43) that there exists a bounded function b such that

$$-\nabla\cdot(A(x)\nabla z) - bz > 0 \text{ in } \mathbb{R}^N. \tag{44}$$

Since $z \geq 0$ and $z(\overline{x}) = 0$, it follows from (44) and from the strong maximum principle that $z \equiv 0$, which is impossible because of the strict inequality in (44).

In the general case, let $(\overline{x}_n) \in \overline{C}$ be such that $x_n - \overline{x}_n \in \prod_{i=1}^N L_i\mathbb{Z}$. Then, up to the extraction of some subsequence, one can assume that there exists $\overline{x}_\infty \in \overline{C}$ such that $\overline{x}_n \to \overline{x}_\infty$ as $n \to +\infty$. Next, set $u_n(x) = u(x + x_n)$, and $p_n(x) = p(x + x_n)$. Since both A and f are periodic with respect to x, the functions u_n and p_n satisfy

$$\begin{array}{l} -\nabla\cdot(A(x+\overline{x_n})\nabla u_n) - f(x+\overline{x_n}, u_n) = 0 \\ -\nabla\cdot(A(x+\overline{x_n})\nabla p_n) - f(x+\overline{x_n}, p_n) = 0 \end{array} \text{ in } \mathbb{R}^N. \tag{45}$$

From standard elliptic estimates, it follows that (up to the extraction of some subsequences) u_n and p_n converge in C^2_{loc} to two functions u_∞ and p_∞ satisfying

$$\begin{array}{l} -\nabla\cdot(A(x+\overline{x_\infty})\nabla u_\infty) - f(x+\overline{x_\infty}, u_\infty) = 0 \\ -\nabla\cdot(A(x+\overline{x_\infty})\nabla p_\infty) - f(x+\overline{x_\infty}, p_\infty) = 0 \end{array} \text{ in } \mathbb{R}^N. \tag{46}$$

Moreover, $u_\infty \geq \varepsilon > 0$ and $p_\infty \geq \varepsilon > 0$.

Let us set $z_\infty(x) := u_\infty(x) - \gamma^* p_\infty(x)$. Then $z_\infty \geq 0$ and $z_\infty(0) = 0$. Furthermore, z_∞ satisfies

$$-\nabla\cdot(A(x+\overline{x_\infty})\nabla z_\infty) - f(x+\overline{x_\infty}, u_\infty(x)) + \gamma^* f(x+\overline{x_\infty}, p_\infty(x)) = 0 \text{ in } \mathbb{R}^N. \tag{47}$$

Then, arguing as for problem (42) above, one obtains a contradiction.

Therefore, we know that $\gamma^* \geq 1$, hence $u \geq p$. By interchanging the roles of u and p, one can prove similarly that $p \geq u$. Furthermore, if p is a positive solution of (7), so is the function $x \mapsto p(x_1, \cdots, x_i + L_i, \cdots, x_N)$, for each $1 \leq i \leq N$. Hence, p is periodic. The proof of Theorem 2 is complete. $\Box$

The same arguments as above lead to the following uniqueness result for a class of solutions of more general elliptic equations with drift terms, under a slightly stronger version of assumption (9) :

Theorem 7 *Let $A = A(x)$ be a symmetric matrix field satisfying (8) and assume that A is of class $C^{1,\alpha}(\mathbb{R}^N)$ and that A and its first-order derivatives are in $L^\infty(\mathbb{R}^N)$. Let B be a vector field of class $C^{0,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$, $(x,s) \mapsto f(x,s)$ be Lipschitz-continuous in s uniformly in x and assume that $f(\cdot, s)$ is of class $C^{0,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ locally in s. Assume that*

$$\forall\ 0 < s < s', \quad \inf_{x \in \mathbb{R}^N} \left(\frac{f(x,s)}{s} - \frac{f(x,s')}{s'} \right) > 0.$$

Let u and v be two positive bounded solutions of

$$\left\{ \begin{array}{l} -\nabla \cdot (A(x)\nabla u) + B(x) \cdot \nabla u = f(x,u) \\ -\nabla \cdot (A(x)\nabla v) + B(x) \cdot \nabla v = f(x,v) \end{array} \right. \quad \text{in } \mathbb{R}^N, \tag{48}$$

such that $\inf_{\mathbb{R}^N} u > 0$ and $\inf_{\mathbb{R}^N} v > 0$.

Then $u = v$.

Remark 6 However, it is not true in general the positive solutions u of (48) are bounded from below by a positive constant under the only assumption $\lambda_1 < 0$, where the generalized first eigenvalue λ_1 is defined as in Remark 5 above.
Indeed, let f be a Lipschitz-continuous function defined in $[0,1]$, such that $f(0) = f(1) = 0$, $f > 0$ on $(0,1)$, $f'(0) > 0$ and $f(s) \leq f'(0)s$ for all $s \in [0,1]$. It is known (see [68]) that, for any $c \geq 2\sqrt{f'(0)}$, there are positive solutions u of

$$u'' - cu' + f(u) = 0, \quad 0 < u < 1 \ \text{ in } \mathbb{R}$$

with $u(-\infty) = 0$ and $u(+\infty) = 1$. But, under the notations of Remark 5, $\lambda_1 = -f'(0) < 0$ in this case.

3.3 Energy of stationary states

This subsection is about an independent result, dealing with the sign of the energy associated to a positive solution of (7), under condition (9). This result, of independent interest, will be used in the forthcoming paper [18] on propagation phenomena.

Assume in this subsection that there exists a positive and bounded solution p of (7) and that condition (9) is fulfilled. It then follows from Theorem 1, part 2), that 0 is an unstable solution of (7) ($\lambda_1 < 0$), and from Theorem 2 that such a function p is then unique and periodic. As we have seen, the existence of p is known for instance if $\lambda_1 < 0$ and if condition (10) is satisfied (from Theorem 1, part 1)).

Consider the energy functional

$$E(u) := \int_C \left\{ \frac{1}{2} \nabla u \cdot (A(x)\nabla u) - F(x,u) \right\} dx, \tag{49}$$

defined on

$$H^1_{per} := \left\{\phi \ \in \ H^1_{loc}(\mathbb{R}^N) \text{ such that } \phi \text{ is periodic}\right\},$$

with $F(x,u) := \displaystyle\int_0^u f(x,s)ds$. We now prove that the energy of p is negative.

Proposition 3 *Assume that condition (9) is satisfied and that there exists a positive bounded solution p of (7). Then $E(p) < 0$.*

Proof. Under the assumptions of Proposition 3, let θ be the function defined in $[0,1]$ by

$$\forall\ t \ \in \ [0,1],\ \theta(t) = E(tp) = \int_C \left\{\frac{1}{2}t^2 \nabla p \cdot (A(x)\nabla p) - F(x, tp(x))\right\} dx. \tag{50}$$

The function θ is of class C^1 and

$$\forall\ t \ \in \ [0,1],\ \theta'(t) = \int_C \left\{t\nabla p \cdot (A(x)\nabla p) - f(x,tp(x))p(x)\right\} dx. \tag{51}$$

From (9) and from the positivity and periodicity of f and p in x, it follows that $f(x,tp(x)) > tf(x,p(x))$ in $\overline{C}$ for all $t \in (0,1)$. Therefore,

$$\forall\ t \ \in \ (0,1),\ \theta'(t) < t\int_C \left\{\nabla p \cdot (A(x)\nabla p) - f(x,p(x))p(x)\right\} dx = 0, \tag{52}$$

the last equality being obtained by multiplication of the equation (7) satisfied by p and integration over C. As a conclusion,

$$E(p) = \theta(1) < \theta(0) = E(0) = 0. \ \square$$

4 The evolution equation

This section is devoted to the

Proof of Theorem 3. Assume that f satisfies (9) and (10). Let u_0 be a nonnegative, not identically equal to 0, bounded and uniformly continuous function, and let $u(t,x)$ be the solution of

$$\begin{cases} u_t - \nabla \cdot (A(x)\nabla u) = f(x,u),\ t \in \mathbb{R}_+,\ x \ \in \ \mathbb{R}^N, \\ u(0,x) = u_0(x),\ x \ \in \ \mathbb{R}^N. \end{cases} \tag{53}$$

Assume first that 0 is an unstable solution of (7) ($\lambda_1 < 0$). Let $\varphi_R = \varphi_R^0$ be the function satisfying (22) with $y = 0$, and call $\lambda_R = \lambda_R^0$. Namely, $\varphi_R \in C^2\left(\overline{B_R}\right)$ and satisfies

$$\begin{cases} -\nabla \cdot (A(x)\nabla\varphi_R) - f_u(x,0)\varphi_R = \lambda_R\varphi_R \text{ in } B_R, \\ \varphi_R > 0 \text{ in } B_R, \quad \varphi_R = 0 \text{ on } \partial B_R,\ \|\varphi_R\|_\infty = 1. \end{cases} \tag{54}$$

From the strong parabolic maximum principle, one has $u(1,x) > 0$ in $\mathbb{R}^N$. Therefore, for $\kappa > 0$ chosen small enough, $\kappa\varphi_R < u(1,x)$ in B_R. Let us extend $\kappa\varphi_R$ to $\mathbb{R}^N$ by setting $v_0(x) := \kappa\varphi_R(x)$ in B_R, and $v_0(x) := 0$ in $\mathbb{R}^N \backslash B_R$. Define the function v_1 by

$$\begin{cases} \partial_t v_1 - \nabla \cdot (A(x)\nabla v_1) = f(x,v_1),\ t \in \mathbb{R}_+,\ x \ \in \ \mathbb{R}^N, \\ v_1(0,x) = v_0(x), \quad \text{for} \quad x \ \in \ \mathbb{R}^N. \end{cases} \tag{55}$$

As it has been done in the course of the proof of Proposition 2, using (40), for R large enough and $\kappa > 0$ small enough, $\kappa\varphi_R$ is a subsolution of (7) in B_R, and therefore v_0 is a "generalized" subsolution of (7) in $\mathbb{R}^N$. Thus v_1 is nondecreasing in time t. Furthermore, $v_1(0,x) \leq u(1,x)$ in $\mathbb{R}^N$ implies

$$v_1(t,x) \leq u(1+t,x) \text{ in } \mathbb{R}_+ \times \mathbb{R}^N. \tag{56}$$

Moreover, for $\kappa > 0$ small enough, $v_0(x) \leq p(x)$ in $\mathbb{R}^N$, where p is the unique positive, and periodic, solution of (7) (the existence and uniqueness of such a p follows from assumptions (9), (10) and $\lambda_1 < 0$, owing to Theorems 2 and 1, part 1)). Since p is a stationary solution of (6), one has

$$v_1(t,x) \leq p(x) \text{ in } \mathbb{R}_+ \times \mathbb{R}^N, \tag{57}$$

Because v_1 is nondecreasing in time t, standard elliptic estimates imply that v_1 converges in $C^2_{loc}\left(\mathbb{R}^N\right)$ to a bounded stationary solution $\underline{v_\infty}(\leq p)$ of (6). Furthermore, one has $\underline{v_\infty}(0) \geq v_1(0,0) \geq \kappa\varphi_R(0) > 0$. Using the strong maximum principle, it follows that $\underline{v_\infty} > 0$ in $\mathbb{R}^N$, and we infer from Theorem 2 that $\underline{v_\infty} \equiv p$.

Next, from (10), there exists $M > 0$ such that $f(x,s) \leq 0$ in $\mathbb{R}^N$ for all $s \geq M$ and $x \in \mathbb{R}^N$. Take M large enough so that $M \geq u_0$ in $\mathbb{R}^N$ and let v_2 be defined by

$$\begin{cases} \partial_t v_2 - \nabla \cdot (A(x)\nabla v_2) = f(x,v_2), \ t \in \ \mathbb{R}_+, \ x \ \in \ \mathbb{R}^N, \\ v_2(0,x) = M, \ x \ \in \ \mathbb{R}^N. \end{cases} \tag{58}$$

Then, since M is a supersolution of (7), v_2 is nonincreasing in time t. Besides, since $v_2(0,x) = M \geq u_0(x) \geq 0$ in $\mathbb{R}^N$,

$$v_2(t,x) \geq u(t,x) \geq 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}^N. \tag{59}$$

Furthermore, since $v_2 \geq 0$ from the maximum principle, v_2 converges in $C^2_{loc}\left(\mathbb{R}^N\right)$ as $t \to +\infty$ to a bounded and nonnegative stationary solution $\overline{v_\infty}(\leq M)$ of (6). From Theorem 2, either $\overline{v_\infty} \equiv 0$ or $\overline{v_\infty} \equiv p$. Finally, one has

$$v_1(t,x) \leq u(1+t,x) \leq v_2(1+t,x), \ t > 0 \ x \ \in \ \mathbb{R}^N. \tag{60}$$

Since $v_1(t,x) \to p(x)$ as $t \to +\infty$, it follows from (60) that $\overline{v_\infty} \equiv p$, and that $u(t,x)$ converges to $p(x)$ in $C^2_{loc}\left(\mathbb{R}^N\right)$ as $t \to +\infty$. Part 1) of Theorem 3 is proved.

Let us now assume that 0 is a stable solution of (7). Then, as carried above, there exists $M > 0$ such that $f(x,s) \leq 0$ for all $s \geq M$ and $x \in \mathbb{R}^N$. Taking M large enough so that $u_0 \leq M$, one again obtains, defining v_2 as above,

$$v_2(t,x) \geq u(t,x) \geq 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}^N. \tag{61}$$

But this time, from the result of Theorem 1, part 2), v_2 converges in $C^2_{loc}\left(\mathbb{R}^N\right)$ to 0 as $t \to +\infty$. Furthermore, the convergence is uniform in x : indeed, v_2 is periodic in x at each time $t \geq 0$, since it is so at $t = 0$, and equation (6) is periodic in x. It follows from (61) that $u(t,x)$ converges to 0 uniformly as $t \to +\infty$, and this concludes the proof of Theorem 3, part 2). □

5 Conservation of species in ecological systems

In this section, we study various effects of the term $f_u(x,0)$ on the principal eigenvalue λ_1 of (11).

5.1 Influence of the “amplitude" of the reaction term

This subsection is devoted to the stability condition of the zero steady state when the nonlinearity f is replaced by Bf in (7), where B is a positive real number. Here, the function f is fixed. Let us call $\lambda_1(Bf)$ the first eigenvalue of (11) with the nonlinearity Bf, and $\phi_B \in C^2\left(\mathbb{R}^N\right)$ the unique principal eigenfunction (with the normalization condition $\|\phi_B\|_\infty = 1$) of

$$\begin{cases} -\nabla \cdot (A(x)\nabla\phi_B) - Bf_u(x,0)\phi_B = \lambda_1(Bf)\phi_B, \\ \phi_B \text{ is positive and periodic in } \mathbb{R}^N. \end{cases} \tag{62}$$

The next two statements are concerned with the dependence of $\lambda_1(Bf)$ with respect to B and correspond to parts 1) and 2) of Theorem 6 respectively.

Proposition 4 *If* $\displaystyle\int_C f_u(x,0) > 0$*, or if* $\displaystyle\int_C f_u(x,0) = 0$ *and* $f_u(x,0) \not\equiv 0$*, then* $\lambda_1(Bf) < 0$ *for every* $B > 0$ *and the function* $B \mapsto \lambda_1(Bf)$ *is decreasing in* $\mathbb{R}^+$*.*

Proof. One first shows that the mapping $B \mapsto \lambda_1(Bf)$ is concave. Since the operator $\mathcal{L}u = -\nabla\cdot(A(x)\nabla u) - Bf_u(x,0)$ is self-adjoint, the eigenvalue $\lambda_1(Bf)$ is obtained from the variational characterization :

$$\lambda_1(Bf) = \min_{\phi\in H^1_{per},\ \phi\not\equiv 0} \frac{\displaystyle\int_C \nabla\phi\cdot(A(x)\nabla\phi) - Bf_u(x,0)\phi^2}{\displaystyle\int_C \phi^2}, \tag{63}$$

where H^1_{per} was defined in the previous section. Thus, it follows that $B \mapsto \lambda_1(Bf)$ is concave, whence continuous (on $\mathbb{R}$).

Next, integrate equation (62) by parts over C. Using the periodicity of ϕ_B, one obtains,

$$-B\int_C f_u(x,0)\phi_B = \lambda_1(Bf)\int_C \phi_B. \tag{64}$$

Take an arbitrary sequence $B_n \to 0$. Since $\lambda_1(B_n f) \to \lambda_1(0) = 0$, standard elliptic estimates and Sobolev injections imply, up to the extraction of some subsequence, that the functions ϕ_{B_n} converge to a nonnegative function ψ, locally (and therefore uniformly by periodicity) in $W^{2,p}$ for all $1 < p < \infty$ (we recall that $f_u(x,0)$ is in L^∞). Furthermore, ψ is such that $\|\psi\|_\infty = 1$, ψ is periodic and satisfies

$$-\nabla\cdot(A(x)\nabla\psi) = \lambda_1(0)\psi = 0. \tag{65}$$

From the strong maximum principle, ψ is positive and $\psi \equiv \phi_0 \equiv 1$. By a classical argument we can then show that the whole family ϕ_B converges to 1 as $B \to 0$.

Then, divide (64) by B and pass to the limit as $B \to 0$, $B \neq 0$. It follows that

$$\left.\frac{d\lambda_1(Bf)}{dB}\right|_{B=0} |C| = -\int_C f_u(x,0), \tag{66}$$

where $|C|$ denotes the Lebesgue measure of C.

Assume now that $\displaystyle\int_C f_u(x,0) > 0$. Since $B \mapsto \lambda_1(Bf)$ is concave, $\lambda_1(0) = 0$ and $\left.\frac{d\lambda_1(Bf)}{dB}\right|_{B=0} < 0$, it follows that $\lambda_1(Bf) < 0$ for every positive B and the function $B \mapsto \lambda_1(Bf)$ is decreasing in $\mathbb{R}^+$.

Similarly, if $\int_C f_u(x,0) = 0$, then $\lambda_1(Bf) \leq 0$ for every positive B. Furthermore, dividing equation (62) by ϕ_B and integrating over C leads to :

$$\lambda_1(Bf)\ |C| = -\int_C \frac{\nabla\phi_B \cdot \nabla(A(x)\nabla\phi_B)}{\phi_B^2}. \tag{67}$$

If $\lambda_1(Bf) = 0$ for some $B > 0$, then ϕ_B is constant, whence $f_u(x,0) \equiv 0$. Therefore, if one further assumes that $f_u(x,0) \not\equiv 0$, then $\lambda_1(Bf) < 0$ for each $B > 0$, and the function $B \mapsto \lambda_1(Bf)$ is decreasing in $\mathbb{R}_+$. This completes the proof of Proposition 4. □

In the case $\int_C f_u(x,0) < 0$, we ow prove the following result.

Proposition 5 *If $\int_C f_u(x,0) < 0$, then $\lambda_1(Bf) > 0$ for all $B > 0$ small enough. If there exists $x_0 \in C$ such that $f_u(x_0,0) > 0$, then, for B large enough, $\lambda_1(Bf) < 0$ and $\lambda_1(Bf)$ is decreasing in B.*

Proof. From the proof of Proposition 4, it is easy to show that, if $\int_C f_u(x,0) < 0$, then $\lambda_1(Bf) > 0$ for $B > 0$ small enough, since $\lambda_1(0) = 0$ and, from (66), $\frac{d\lambda_1(Bf)}{dB}\Big|_{B=0} = -\int_C f_u(x,0) > 0$.

There exists a positive and periodic function ϕ_0 such that

$$\int_C f_u(x,0)\phi_0^2 > 0. \tag{68}$$

Then, from (63),

$$\lambda_1(Bf) \leq \frac{\displaystyle\int_C \left[\nabla\phi_0 \cdot (A(x)\nabla\phi_0) - Bf_u(x,0)\phi_0^2\right] dx}{\displaystyle\int_C \phi_0^2}. \tag{69}$$

Clearly, this shows that $\lambda_1(Bf) < 0$ for B large enough. The concavity of $B \mapsto \lambda_1(Bf)$ and the fact that $\lambda_1(0) = 0$ then imply that $B \mapsto \lambda_1(Bf)$ is decreasing at least when $\lambda_1(Bf)$ is negative, and thus for $B > 0$ large enough. □

5.2 Influence of the “shape" of $f_u(x,0)$

This section is concerned with the study of the dependence of the first eigenvalue λ_1 of (11) on the shape of the function $f_u(x,0)$. One denotes $\mu(x) = f_u(x,0)$ and $\lambda_1 = \lambda_1[\mu]$. The following proposition compares the effect of μ and of its average.

Proposition 6 *Let μ_0 be a real number. Then*

$$\lambda_1[\mu] \leq \lambda_1[\mu_0], \tag{70}$$

as soon as $\int_C \mu = \mu_0|C|$ (where $|C|$ is the Lebesgue measure of the set C).

Proof. From (11), replacing $f_u(x,0)$ by $\mu(x)$, one obtains

$$-\nabla \cdot (A(x)\nabla\phi) - \mu(x)\phi = \lambda_1\phi = \lambda_1[\mu]\phi, \quad x \in \mathbb{R}^N, \tag{71}$$

where $\phi > 0$ is the principal periodic eigenfunction associated to λ_1, with the normalization condition $\|\phi\|_\infty = 1$. Dividing (71) by ϕ and integrating by parts over C yields

$$-\int_C \frac{\nabla\phi \cdot (A(x)\nabla\phi)}{\phi^2} - \int_C \mu = \lambda_1|C| = \lambda_1[\mu]|C|. \tag{72}$$

Clearly, clearly, $\phi_{\mu_0} \equiv 1$ and $\lambda_1[\mu_0] = -\mu_0$. Therefore, it follows from equation (72) that

$$\lambda_1[\mu] \leq -\frac{\int_C \mu}{|C|} = -\mu_0 = \lambda_1[\mu_0].$$

This completes the proof of Proposition 6. □

In a sense, Proposition 6 shows that a heterogeneous medium improves the "biological conservation", in comparison with a constant medium μ_0.

Moreover, the medium becomes all the more conservative the more it is heterogeneous. This statement is made precise in the following

Proposition 7 *Let $\mu_0 \in \mathbb{R}$ and let f be such that $f_u(x,0) = \mu(x) = \mu_0 + B\nu(x)$, where ν has zero average and $\nu \not\equiv 0$. Let $\lambda_{1,B} = \lambda_1[\mu]$ be the first eigenvalue of (71). Then the function $B \mapsto \lambda_{1,B}$ is decreasing in $\mathbb{R}_+$. Furthermore, $\lambda_{1,B}$ is negative for all $B > 0$ if $\mu_0 \geq 0$, and $\lambda_{1,B}$ is negative for $B > 0$ large enough if $\mu_0 < 0$.*

Proof. As in Proposition 6, it can be shown that the function $B \mapsto \lambda_{1,B}$ is concave, and $\frac{d\lambda_{1,B}}{dB}\Big|_{B=0} = 0$, $\lambda_{1,0} = -\mu_0$. The conclusion follows as in the proofs of Propositions 5 and 6. □

Let us now turn out to the effect of rearranging the level sets of μ. We denote by μ^* the function obtained by performing a succession of Steiner periodic rearrangement of μ with respect to the ordered variables $x_1, ..., x_N$ (see Section 2.3.1 above for the definition) .

Proposition 8 *Under the above notations, and assuming furthermore that A is the identity matrix, the following inequality holds*

$$\lambda_1[\mu^*] \leq \lambda_1[\mu]. \tag{73}$$

Proof. The proof rests on rearrangement inequalities. Let k be a nonnegative real number such that $\mu + k \geq 0$ in $\mathbb{R}^N$, and let ϕ be the principal eigenfunction associated to $\lambda_1[\mu]$, with the normalization condition $\|\phi\|_\infty = 1$.

A classical inequality for rearrangement (Compare e.g. [65]) asserts that :

$$\int_C (\mu + k)^*(\phi^*)^2 \geq \int_C (\mu + k)\phi^2. \tag{74}$$

Since $(\mu+k)^* = \mu^* + k$, one infers from (74) that $\int_C \mu^*(\phi^*)^2 \geq \int_C \mu\phi^2 + k\int_C [\phi^2 - (\phi^*)^2]$. On the other hand, $\int_C [\phi^2 - (\phi^*)^2] = 0$, whence

$$\int_C \mu^*(\phi^*)^2 \geq \int_C \mu\phi^2. \tag{75}$$

Next, it follows from Theorem 2.1 and Remark 2.6 in [65] (see also [12], [20]) that

$$\int_C |\nabla\phi|^2 \geq \int_C |\nabla\phi^*|^2. \tag{76}$$

As already emphasized, $\lambda_1[\mu^*]$ and $\lambda_1[\mu]$ are given by the following variational formulæ

$$\lambda_1[\mu] = \min_{\psi\in H^1_{per},\ \psi\not\equiv 0} \frac{\int_C \left(|\nabla\psi|^2 - \mu\psi^2\right)}{\int_C \psi^2}, \tag{77}$$

and

$$\lambda_1[\mu^*] = \min_{\psi\in H^1_{per},\ \psi\not\equiv 0} \frac{\int_C \left(|\nabla\psi|^2 - \mu^*\psi^2\right)}{\int_C \psi^2}. \tag{78}$$

Furthermore, the minimum in (77) is reached for $\psi = \phi$. It follows from (78) that

$$\lambda_1[\mu^*] \leq \frac{\int_C \left(|\nabla\phi^*|^2 - \mu^*(\phi^*)^2\right)}{\int_C (\phi^*)^2}. \tag{79}$$

From (75), one has $\int_C \mu^*(\phi^*)^2 \geq \int_C \mu\phi^2$, and, from (76), $\int_C |\nabla\phi|^2 \geq \int_C |\nabla\phi^*|^2$. One also knows that $\int_C (\phi)^2 = \int_C (\phi^*)^2$. Finally, it follows from (79) that

$$\lambda_1[\mu^*] \leq \frac{\int_C \left(|\nabla\phi|^2 - \mu\phi^2\right)}{\int_C \phi^2} = \lambda_1[\mu], \tag{80}$$

and Proposition 8 is proved. □

As a conclusion, one can say that from the biological conservation standpoint, among all periodic $\overline{\mu}$ having a given distribution function, the optimal one is necessarily Steiner symmetric, that is, symmetric with respect to $x_i = 0$ and decreasing in x_i, for $x_i \in [0, L_i/2]$ (for each $i = 1, \cdots, N$). Note, however, that the actual optimal shape (among all Steiner symmetric functions in all variables) is not known, even when μ takes only two values.

6 The effect of fragmentation in bounded domain models

In this section, we summarize some properties concerned with the case of bounded domains, and, using a symmetrization argument, we state some general results extending previous work .

Consider the equation

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \Omega, \tag{81}$$

set in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Assume, say, that the nonlinearity f is smooth and satisfies (9-10), and that A is a smooth uniformly elliptic matrix field.

For Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega,$$

there exists a solution p of

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) = f(x,p), & x \in \Omega \\ p(x) > 0, & x \in \Omega \\ p(x) = 0, & x \in \partial\Omega, \end{cases} \tag{82}$$

if and only if the first eigenvalue λ_1 of the operator $\mathcal{L}\phi = -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi$ in Ω (with Dirichlet boundary conditions on $\partial\Omega$) is negative. Furthermore, if it exists, p is unique.

In the one-dimensional case with constant diffusion du_{xx} and f of the type $f(u) = u - \alpha u^2 - \beta u^2/(1+u^2)$, this result is due to Ludwig, Aronson and Weinberger [70] (see also Murray and Sperb [73] for the two-dimensional case with additional drift terms). It was generalized to any dimension by Cantrell and Cosner [27], in the case of a nonlinearity f of the type $f(x,u) = m(x)u - c(x)u^2$ (with $c(x) > 0$) and with constant diffusion $d\Delta u$.

For the general equation (82), the results mentioned above can be proved with the same methods as the ones used in the present paper. Notice that the case of bounded domains is actually much simpler than the periodic case in $\mathbb{R}^N$. In particular, uniqueness of the positive solutions, in particular, was proved in [10].

Furthermore, using the same arguments as those of section 2.2, the solutions $u(t,x)$ of (81) with initial condition $u_0 \geq 0$, $u_0 \not\equiv 0$ converge uniformly in $x \in \overline{\Omega}$ as $t \to +\infty$ to the unique positive solution $p(x)$ if $\lambda_1 < 0$; otherwise, that is if $\lambda_1 \geq 0$, then $u(t,x) \to 0$ uniformly in $x \in \overline{\Omega}$ as $t \to +\infty$ (see also [27, 28, 70] for earlier results in some particular cases).

Some of the above results have been extended by Cantrell and Cosner [29] to some special cases of systems of two equations.

For problem (81) in a bounded interval $(0,L)$, the influence of the location of the favourable and unfavourable regions has been studied in [28], on the basis of explicit analytic calculations. This work is restricted to the case of the patch model, that is when the birth rate $f_u(x,0)$ is piecewise constant and only takes two possible values. For Dirichlet boundary conditions, it is better for species conservation to have the most favourable region concentrated around the middle of the interval, away from the boundary. On the contrary, in the case of Neumann boundary conditions, it is better for species conservation to have the favourable and unfavourable regions concentrated near each of the two boundary points of the interval.

Using symmetrization techniques as we did above for the periodic case allows us to extend and much simplify these results. For a general $\mu(x)$, we prove the following :

Theorem 8 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and assume that Ω is convex in some direction say x_1, and symmetric with respect to $x_1 = 0$. Let μ be continuous in $\overline{\Omega}$ and let $\lambda_1[\mu]$ be the first eigenvalue of*

$$\begin{cases} -\Delta\varphi - \mu(x)\varphi = \lambda_1[\mu]\varphi & in \quad \Omega \\ \varphi = 0 \quad on \quad \partial\Omega. \end{cases}$$

Then, $\lambda_1[\mu^*] \leq \lambda_1[\mu]$, where μ^* is the Steiner symmetrization of μ in the variable x_1, with respect to $\{x_1 = 0\}$ and nonincreasing away from $\{x_1 = 0\}$.

Furthermore, equality $\lambda_1[\mu] = \lambda_1[\mu^*]$ holds if and only if μ is symmetric with respect to x_1 and nonincreasing away from $\{x_1 = 0\}$.

In an interval, this theorem provides the optimal rearrangement of a function μ in the sense of finding, among all μ having a given distribution function that function, namely μ^*, that minimizes $\lambda_1[\mu]$. In higher dimensions, this is not known. In dimension $N = 2$ consider the simple patch model when μ is allowed to take two values. When Ω is say a rectangle, then the shape of the optimal μ is not known. From the theorem, we know that it is doubly symmetric in the Steiner sense, hence the favourable is connected. However, it is not known that this region is convex, a property which we conjecture holds.

For Neumann boundary conditions, the situation is more delicate. One has to use *monotone rearrangement* (see [65, 20] for definition and properties and [12] for the pecise inequality). Consider now the eigenvalue problem :

$$\begin{cases} -\Delta\varphi - \mu(x)\varphi = \Lambda_1[\mu]\varphi & \text{in} \quad \Omega \\ \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Like for the periodic case or for the bounded domain case with Dirichlet condition, the sign of $\Lambda_1[\mu]$ determines the existence of stationary solutions and asymptotic bahaviour of the solution of the nonlinear problem (81) with Neumann condition.

The monotone rearrangement of a function $v(x)$ of one variable, on an interval (a, b), is defined as the unique monotone (say) nondecreasing function $v^\sharp$ on (a, b) which has the same distribution function as v. Then, define the Steiner monotone rearrangement of a function $v(x_1, \ldots, x_N)$ on a set $\{x \,;x_i \in (a_i, b_i),\ \forall i = 1, \ldots, N\}$, as the function $v^\sharp$ which is obtained from v by performing successive monotone Steiner rearrangements in each of the directions $x_1, \ldots, x_N$.

Theorem 9 *Assume that Ω is a cube $\{x \,;x_i \in (a_i, b_i),\ \forall i = 1, \ldots, N\}$. Under Steiner monotone rearrangement, the Neumann eigenvalue satisfies the following inequality : $\Lambda_1[\mu^\sharp] \leq \Lambda_1[\mu]$.*

This theorem rests on the following rearrangement inequality :

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega |\nabla\varphi^\sharp|^2. \tag{83}$$

This is well known in dimension 1 (see [65]) but somewhat delicate in dimension N. This inequality is proved in [12].

As a consequence of this result, we see that, in the simplified case of the patch model on a rectangle, the rearranged configuration, where all the favourable patch is concentrated in one of the corners of the domain, leaves better chances of survival than an originally fragmented configuration. An example is given in the figure 4 below.

Remark 7 As a consequence of this theorem, in the patch case, the optimal rearrangement, under the constraint of a given area of the unfavourable zone, is a μ such that $\mu^\sharp = \mu$. However, like for the other cases, in higher dimension, the question of the optimal shape of the environment (in the patch model) is still open. This appears to be an interesting mathematical question.

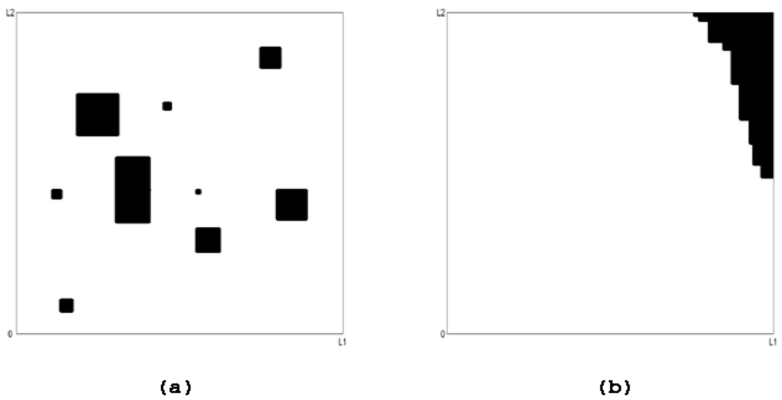

FIG. 4 – (a) Initial patch, and (b) after monotone Steiner rearrangement

Chapitre 2 : Biological invasions and pulsating travelling fronts

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Sommaire

1 Introduction and main results

This paper deals with the mathematical analysis of a periodically fragmented environment model which is given by the reaction-diffusion equation

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \mathbb{R}^N \tag{1}$$

with periodic dependence in the x variables. It is the sequel to the paper [17], which focused on some conditions for the stationary equation

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) = f(x,p) \ \text{ in } \mathbb{R}^N, \\ p(x) > 0, \ x \in \mathbb{R}^N, \end{cases} \tag{2}$$

to have a bounded solution, and on the effects of the heterogeneity in x. The present paper is concerned with the propagation pheneomena, and especially the propagation of fronts, associated to (1) – precise definitions will be given below. Some formulas for the speeds of propagation of fronts are proved and the dependence in terms of the coefficients of (1) is analyzed.

The archetype of such reaction-diffusion models is the following equation

$$u_t - \Delta u = f(u) \ \text{ in } \mathbb{R}^N \tag{3}$$

which was introduced in the pioneering papers of Fisher [43] and Kolmogorov, Petrovsky and Piscounov [68]. An example of nonlinear term is given by the logistic law $f(u) = u(1-u)$. This type of equation was first motivated by population genetics, and, as (1), it also arises in more general models for biological invasions or combustion.

Of particular interest are the propagation phenomena related to reaction-diffusion equations of the type (3), or (1). First, equation (3) may exhibit planar travelling fronts, which are special solutions of the type $u(t,x) = U(x \cdot e + ct)$ for some direction e ($|e| = 1$, $-e$ is the direction of propagation) and $U : \mathbb{R} \to (0,1)$ (assuming that $f(0) = f(1) = 0$). Such solutions are invariant in time in the comoving frame with speed c in the direction $-e$. Second, starting with an initial datum $u_0 \geq 0$, $\not\equiv 0$ which vanishes outside some compact set, then, under some assumptions on f, $u(t,x) \to 1$ as $t \to +\infty$; furthermore, the set where u is close to 1 expands at a certain speed which is the asymptotic speed of spreading and which, in the case of equation (3) with a nonlinearity f positive in $(0,1)$, is the minimal speed of planar fronts (see *e.g.* [5]).

Whereas the homogeneous equation (3) has attracted many works in the mathematical literature, propagation phenomena for *heterogeneous* equations of the type (1), where both the diffusion and the reaction coefficients depend on the space variables x, were studied more recently (see *e.g.* [13, 44, 56, 85, 93]). In models of biological invasions, the heterogeneity may be a consequence of the presence of highly differentiated zones, such as forests, fields, roads, cities, etc., where the species in consideration may tend to diffuse, reproduce or die with different rates from one place to another.

One focuses here on *periodic* environments models and for which the diffusion matrix $A(x)$ and the reaction term $f(x,u)$ now depend on the variables $x = (x_1, \cdots, x_N)$ in a periodic fashion. As an example, f may be of the type

$$f(x,u) = u(\mu(x) - \kappa(x)u), \tag{4}$$

or even, simply,

$$f(x,u) = u(\mu(x) - u), \tag{5}$$

where the periodic coefficient $\mu(x)$, which may well be negative, can be interpreted as an effective birth rate of the population and the periodic function $\kappa(x)$ reflects a saturation effect related to competition for resources. The lower μ is, the less favorable the environment is to the species.

These models for biological invasions in unbounded domains were first introduced by Shigesada et al. in dimensions 1 and 2 (see [66, 85, 86]). In these works, the nonlinearity f is given by (5), and A and μ are piecewise constant and only take two values. This model is then referred to as the patch model. Numerical simulations and formal arguments were discussed in [66, 85, 86] about this model – in space dimensions 1 or 2. The various works of Shigesada and her collaborators have been an inspiring source for the present paper. We aim here at proving rigorously some properties which had been discussed formally or observed numerically. The introduction of new mathematical ideas will furthermore allow us to derive results in greater generality and for higher dimensional problems as well.

In the paper [17] and in the present one, we discuss these types of problems in the framework of a general periodic environment, and we give a complete and rigorous mathematical treatment of these questions. In the first paper [17], we discussed the existence of a positive stationary state of (1), that is a positive bounded solution p of (2). The latter is referred to as biological conservation. We also analyzed in [17] the effects of fragmentation of the medium and the effects of coefficients with large amplitude on biological conservation. Here, we connect the condition for species survival (existence of such a solution p) to that for propagation of pulsating fronts for which the heterogeneous state p invades the uniform state 0. This type of question is referred to as biological invasion. We also analyze the effects of the heterogeneity of the medium on the speed of propagation. We especially prove a monotonous dependence of the speed of invasion on the amplitude of the effective birth rate.

Let us make the mathematical assumptions more precise. Let $L_1, \cdots, L_N > 0$ be N given positive real numbers. A function $g : \mathbb{R}^N \to \mathbb{R}$ is meant to be periodic if $g(x_1, \cdots, x_k + L_k, \cdots, x_N) \equiv g(x_1, \cdots, x_N)$ for all $k = 1, \cdots, N$. Let C be the period cell defined by

$$C = (0, L_1) \times \cdots \times (0, L_N).$$

The diffusion matrix field $A(x) = (a_{ij}(x))_{1\leq i,j\leq N}$ is assumed to be symmetric ($a_{ij} = a_{ji}$), periodic, of class $C^{2,\alpha}$ (with $\alpha > 0$),[4] and uniformly elliptic, in the sense that

$$\exists \alpha_0 > 0,\ \forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N, \qquad \sum_{1\leq i,j\leq N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2. \tag{6}$$

The function $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is of class $C^{1,\alpha}$ in (x, u) and C^2 in u, periodic with respect to x. One assumes that $f(x, 0) = 0$ for all $x \in \mathbb{R}^N$ and one sets $f_u(x, 0) := \lim_{s\to 0^+} f(x, s)/s$. Furthermore, throughout the paper, one assumes that

$$\forall x \in \mathbb{R}^N, \quad s \mapsto f(x, s)/s \text{ is decreasing in } s > 0 \tag{7}$$

and

$$\exists M \geq 0,\ \forall s \geq M,\ \forall x \in \mathbb{R}^N, \quad f(x, s) \leq 0. \tag{8}$$

[4] The smoothness assumptions on A, as well as on f below, are made to ensure the applicability of some a priori gradients estimates for the solutions of some approximated elliptic equations obtained from (1) (see Lemma 6 in Section 2.3). These gradient estimates are obtained for smooth (C^3) solutions through a Bernstein-type method, [14]. We however believe that the smoothness assumptions on A, as well as on f, could be relaxed, by approximating A and f by smoother coefficients.

Examples of functions f satisfying (7-8) are functions of the type (4) or (5), namely $f(x,u) = u(\mu(x) - \kappa(x)u)$ or simply $f(x,u) = u(\mu(x) - u)$, where μ and κ are $C^{1,\alpha}$ periodic functions.

Let λ_1 be the principal eigenvalue of the operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0\phi := -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi,$$

with periodicity conditions. Namely, λ_1 is the unique real number such that there exists a C^2 function $\phi > 0$ which satisfies

$$\begin{cases} -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1\phi \text{ in } \mathbb{R}^N, \\ \phi \text{ is periodic, } \ \phi > 0. \end{cases} \tag{9}$$

One says that 0 is an unstable solution of (2) if $\lambda_1 < 0$, and "stable" if $\lambda_1 \geq 0$.

We especially proved in [17] that, if $\lambda_1 \geq 0$, then 0 is the only nonnegative bounded solution of (2) and any solution of (1) with bounded nonnegative initial condition u_0 converges to 0 uniformly in $x \in \mathbb{R}^N$ as $t \to +\infty$ (one refers to this phenomenon as extinction). On the other hand, if $\lambda_1 < 0$, then there is a unique positive bounded solution p of (2), which turns out to be periodic,[5] and the solution $u(t,x)$ converges to $p(x)$ locally in x as $t \to +\infty$, as soon as $u_0 \geq 0, \not\equiv 0$.

The above results motivate the following

Definition 1 *We say that the* hypothesis for conservation *is satisfied if there exists a positive bounded solution p of (2).*

A simple necessary and sufficient condition for the hypothesis for conservation (or survival) to be satisfied is that $\lambda_1 < 0$, and the solution p is then unique and periodic. This hypothesis is fulfilled especially if $f_u(x,0) \geq 0, \not\equiv 0$. An example of a function f satisfying the hypothesis is the classical Fisher-KPP nonlinearity $f(x,u) = f(u) = u(1-u)$ (see [43, 68]), where $p(x) \equiv 1$. For a general nonlinearity f satisfying (7) and (8), comparison results and conditions on $f_u(x,0)$ for λ_1 to be negative are given in [17] (see also Theorem 2 below). However, it is not easy to understand in general the interaction between the heterogeneous diffusion and reaction terms.

One focuses here on the set of solutions which describe the invasion of the uniform state 0 by the periodic positive function p, when the hypothesis for conservation is satisfied. A solution $u(t,x)$ of (1) is called a *pulsating travelling front* propagating in the direction $-e$ with the effective speed $c \neq 0$ if

$$\begin{cases} \forall\ (t,x) \ \in \ \mathbb{R} \times \mathbb{R}^N, \ u_t - \nabla \cdot (A(x)\nabla u) \ = \ f(x,u), \\ \forall\ k \ \in \ \displaystyle\prod_{i=1}^{N} L_i\mathbb{Z}, \ \forall\ x \ \in \ \mathbb{R}^N, \ u\left(t + \frac{k \cdot e}{c}, x\right) \ = \ u(t, x+k), \end{cases} \tag{10}$$

with the asymptotic conditions

$$u(t,x) \underset{x\cdot e \to -\infty}{\to} 0, \ u(t,x) - p(x) \underset{x\cdot e \to +\infty}{\to} 0. \tag{11}$$

The above limits are understood as local in t, and uniform in the directions of $\mathbb{R}^N$ orthogonal to e.

Our first result is the following existence theorem :

[5] Notice that the periodicity is forced by the uniqueness, but was not a priori required in the formulation of equation (2).

Theorem 1 *Under the above assumptions on A and f, and under the hypothesis for conservation, there exists $c^* > 0$ such that problem (10-11) has a classical solution (c, u) if and only if $c \geq c^*$. Furthermore, any such solution u is increasing in the variable t.*

Lastly, the minimal speed c^* is given by the following variational formula

$$c^* = \min \{c,\ \exists\ \lambda > 0 \text{ such that } \mu_c(\lambda) = 0\},$$

where $\mu_c(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{aligned} -L_{c,\lambda}\psi &= -\nabla \cdot (A(x)\nabla\psi) - 2\lambda\ eA(x)\nabla\psi \\ &\quad -[\lambda\nabla \cdot (A(x)e) + \lambda^2 eA(x)e - \lambda c + f_u(x,0)]\psi, \end{aligned} \tag{12}$$

with periodicity conditions.

Before going further on, let us briefly comment this result and recall some earlier works in the literature. Observe first that the formula for the minimal speed simply reduces to the well-known Fisher KPP formula $2\sqrt{f'(0)}$ for the minimal speed of planar front $\phi(x \cdot e + ct)$ for the homogeneous equation $u_t - \Delta u = f(u)$ in $\mathbb{R}^N$ with f satisfying (7-8) and $p(x) \equiv \min\{s > 0,\ f(s) \leq 0\}$. Periodic nonlinearities $f(x, u)$ in space dimension 1 were first considered by Shigesada, Kawasaki and Teramoto [86], and by Hudson and Zinner [56].[6] The case of equations $u_t - \Delta u + v \cdot \nabla u = f(u)$ with shear flows $v = (\alpha(y), 0, \cdots, 0)$ in straight infinite cylinders $\{(x_1, y) \in \mathbb{R} \times \omega\}$ was dealt with by Berestycki and Nirenberg [24], under the assumption that f stays positive in, say, $(0, 1)$; min-max type formulas for c^* were obtained in [48]. Berestycki and Hamel [13] generalized the notion of pulsating fronts and got existence and monotonicity results in the framework of more general periodic equations $u_t - \nabla \cdot (A(x)\nabla u) + v(x) \cdot \nabla u = f(x, u)$ in periodic domains, under the assumption that $f \geq 0$ and $f(x, 0) = f(x, 1)$, $f(x, s) > 0$ for all $s \in (0, 1)$. A formula for the minimal speed is given in [16] under the assumption that $f(x, s) \leq f_u(x, 0)s$ for all $s \in [0, 1]$ and the dependence of c^* in terms of the diffusion, advection, reaction coefficients as well as the geometry of the domain, is analyzed. Some lower and upper bounds for the minimal speed when the advection term v is large are given in [6, 11, 15, 33, 52, 67]. Lastly, let us add that some previously mentionned works, as well as other ones, [5, 13, 22, 24, 41, 42, 48, 51, 52, 53, 65, 91, 93], were also devoted to other types of nonlinearities (combustion, bistable), for which the speed of propagation of fronts may be unique.

One of the difficulties and specificities of problem (10-11) with a nonlinearity f satisfying (7-8) is that f may now be negative at some points x, whereas it is positive at other places, for the same value of u. Besides the existence of pulsating fronts and the variational characterization of the minimal speed, Theorem 1 above also gives the monotonicity of all fronts in the variable t (notice that a similar formula for c^* was given by Weinberger in [92], with a different approach, but the monotonicity of the front was a priori assumed there).

Consider now a nonlinearity f satisfying (7-8), and such that $f_u(x, 0) = \mu(x) + B\nu(x)$, where μ and ν are given periodic functions and B is a positive parameter. If $\lambda_1 < 0$ (namely if the hypothesis for conservation is satisfied), one calls $c^*(B)$ the minimal speed, given in Theorem 1, of the pulsating fronts solving (10-11). The following theorem especially gives a monotonous dependency of $c^*(B)$ on B as well as some lower and upper bounds for large or small B (when $\mu \equiv 0$, B can then be viewed as the amplitude of the effective birth rate of the species in consideration). Furthermore, Theorem 2 below also deals with the influence of the heterogeneity of f on the minimal speed of pulsating fronts.

[6] Hudson and Zinner proved the existence of one-dimensional pulsating fronts for problems of the type $u_t - u_{xx} = f(x, u)$, provided $c \geq c^*$, but did not actually prove that c^* was the *minimal* speed.

Theorem 2 *Assume that A is a constant symmetric positive matrix and assume that f satisfies (7-8) and that $f_u(x,0)$ is of the type $f_u(x,0) = \mu(x) + B\nu(x)$, where μ and ν are given periodic $C^{1,\alpha}$ functions, and $B \in \mathbb{R}$.*

a) Assume that $\max\ \nu > 0$. Then the hypothesis for conservation ($\lambda_1 < 0$) is satisfied for $B > 0$ large enough and

$$c^*(B) \le 2\sqrt{eAe \max(\mu + B\nu)}\ ;$$

furthermore, if $\mu = \mu_0$ is constant, then $c^*(B)$ is increasing in B (for B large enough so that $\lambda_1 < 0$).

b) Assume that $\int_C \mu \ge 0$, $\int_C \nu \ge 0$ and $\max\ \nu > 0$. Then the hypothesis for conservation is satisfied for all $B > 0$, and $c^*(B)$ is increasing in $B > 0$ under the additional assumption that $\mu = \mu_0 \ge 0$ is constant. Furthermore, for all $B > 0$,

$$2\sqrt{\frac{eAe}{|C|}\int_C (B^{-1}\mu(x) + \nu(x))dx} \le \frac{c^*(B)}{\sqrt{B}} \le 2\sqrt{eAe\ \max(B^{-1}\mu + \nu)}$$

and

$$\frac{1}{2}\sqrt{eAe\ \max\ \nu} \le \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \le \limsup_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \le 2\sqrt{eAe\ \max\ \nu}.$$

c) Assume that $\mu \equiv 0$, $f_u(x,0) = B\nu(x)$ with $\int_C \nu \ge 0$, $\max\ \nu > 0$. One has

$$\lim_{B\to 0^+} \frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx}.$$

Theorem 2 implies especially that, when $f_u(x,0)$ is of the type $\mu(x) + B\nu(x)$, and no matter how bad the environment may be elsewhere, it suffices to have a very favorable (even quite narrow) zone (namely $\nu > 0$ somewhere) to allow for species survival and to increase the speed of propagation of fronts. Furthermore, the speed is comparable to $\sqrt{B}$ for large amplitudes B as soon as, say, $\int_C \nu > 0$.

Lastly, call $c^*[\mu]$ the minimal speed of pulsating travelling fronts solving he problem (10-11) with $f_u(x,0) = \mu(x)$, provided the assumption for conservation is satisfied. From the previous theorem, one immediately deduces the following corollary :

Corollary 1 *Assume that A is a constant symmetric positive matrix and assume that f satisfies (7-8), with $f_u(x,0) = \mu(x)$. Assume that $\int_C \mu \ge \mu_0|C|$ with $\mu_0 > 0$. Then f satisfies the hypothesis for conservation and*

$$c^*[\mu] \ge c^*[\mu_0] = 2\sqrt{(eAe)\mu_0}.$$

This corollary simply means that the heterogeneity of the medium increases the speed of propagation of pulsating fronts, in any given unit direction of $\mathbb{R}^N$.

As already underlined, the main difference with the results in [13], in the existence and monotonicity result (Theorem 1), is that the function f here is not assumed to be nonnegative. The nonnegativity of f played a crucial role in [13], where the existence of the minimal speed c^* was proved by approximating f with cut-off functions, as in [24]. Although we solve some regularized problems in bounded domains as in [13], the method used in this paper is rather different since we directly prove that the set of possible speeds c is an interval which is not bounded from above, and we define c^* as the minimum of this interval. Existence of pulsating fronts is proved in Section 2. Monotonicity is proved in Section 3. Lastly, the characterization of c^* is given in Section 4, as well as the effects of the heterogeneity of the medium on the propagation speeds.

2 Existence result

This section is devoted to the proof of the existence of pulsating fronts for (10-11) for large speed. Throughout this section, one assumes that the hypothesis for conservation is satisfied, namely that there exists a (unique) positive bounded solution p of (2), which is periodic.

2.1 Existence result in finite cylinders for a regularized problem

Let us make the same change of variables as Xin [93] and Berestycki, Hamel [13]. Let $\phi(s,x)$ be the function defined by :

$$\phi(s,x) = u\left(\frac{s - x \cdot e}{c}, x\right)$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where u is a classical solution of (10-11).

The function ϕ satisfies the following degenerate elliptic equation

$$\begin{aligned} \nabla_x \cdot (A(x)\nabla_x \phi) + (eA(x)e)\phi_{ss} + \nabla_x \cdot (A(x)e\phi_s) & \\ +\partial_s(eA(x)\nabla_x\phi) - c\phi_s + f(x,\phi) &= 0 \quad \text{in } \mathcal{D}'_L(\mathbb{R}\times\mathbb{R}^N) \end{aligned} \tag{13}$$

together with the periodicity condition

$$\phi \text{ is L-periodic with respect to } x. \tag{14}$$

Moreover, since $u(t,x) \to 0$ as $x \cdot e \to -\infty$ and $u(t,x) - p(x) \to 0$ as $x \cdot e \to +\infty$, locally in t and uniformly in the directions of $\mathbb{R}^N$ which are orthogonal to e, and since ϕ is L-periodic with respect to x, one gets

$$\phi(-\infty,x) = 0, \quad \phi(+\infty,x) = p(x) \text{ uniformly in } x \in \mathbb{R}^N. \tag{15}$$

Conversely, if ϕ is a solution of (13-15) such that $u(t,x) = \phi(x \cdot e + ct, x)$ is C^1 in t, C^2 in x, then u is a classical solution of (10-11).

Let a and ε be two positive real numbers, and set $\Sigma_a = (-a,a)\times\mathbb{R}^N$. As it was done in [13], one first works with elliptic regularizations of (13) of the type

$$\left\{ \begin{array}{rcl} L_\varepsilon\phi + f(x,\phi) &=& 0 \text{ in } \Sigma_a, \\ \phi \text{ is L-periodic w.r.t. } x, && \\ \forall x \in \mathbb{R}^N, \quad \phi(-a,.) = 0,\ \phi(a,x) &=& p(x), \end{array} \right. \tag{16}$$

where L_ε is the elliptic (in the (s,x)-variables) operator defined by

$$\begin{aligned} L_\varepsilon\phi \quad = \quad & \nabla_x \cdot (A(x)\nabla_x\phi) + (eA(x)e + \varepsilon)\phi_{ss} \\ & + \nabla_x \cdot (A(x)e\phi_s) + \partial_s(eA(x)\nabla_x\phi) - c\phi_s. \end{aligned}$$

We will follow the scheme as in [13] to prove the existence of solutions of (16) and state some of their properties, only indicating the differences which may appear.

Let us establish at first the

Lemma 1 *For each $c \in \mathbb{R}$, there exists a solution $\phi^c \in C^2\left(\overline{\Sigma_a}\right)$ of (16).*

PROOF. Let ψ be the function defined by $\psi(s,x) = p(x)\dfrac{s+a}{2a}$. One sets $v = \phi - \psi$. Then, since p satisfies $-\nabla\cdot(A(x)\nabla p) - f(x,p) = 0$ in $\mathbb{R}^N$, the problem (16) is equivalent to

$$\left\{\begin{array}{rcl} -L_\varepsilon v & = & f(x, v+\psi) - \dfrac{s+a}{2a} f(x,p) \\ & & +(2a)^{-1}[2A(x)e\cdot\nabla p + \nabla\cdot(A(x)e)\ p - cp] \ \text{ in } \Sigma_a, \\ & & v \text{ is L-periodic w.r.t. } x, \\ v(-a,.) & = & 0, \quad v(a,.) \ = \ 0. \end{array}\right. \tag{17}$$

Using the fact that $f(x,p)$, p, A, ∇p and $\nabla_x\cdot(Ae)$ are globally bounded, (since p and A are L-periodic and C^1), one can follow the proof of Lemma 5.1 of [13]. Namely, using Lax-Milgram Theorem with Schauder fixed point Theorem, one can find a solution v of the first equation of (17), in the distribution sense, in Σ_a. Then, from the regularity theory up to the boundary, this solution v is a classical solution of (17) in $\overline{\Sigma_a}$. Finally, the function $\phi = v + \psi \in C^2\left(\overline{\Sigma_a}\right)$ is a classical solution of (16). □

Lemma 2 *The function ϕ^c defined above is increasing in s and it is the unique solution of* (16) *in $C^2\left(\overline{\Sigma_a}\right)$.*

PROOF. One has to show at first that $0 < \phi^c(s,x) < p(x)$ in Σ_a. Since $f \equiv 0$ in $\mathbb{R}^N \times (-\infty, 0]$, the strong elliptic maximum principle yields that $\phi^c > 0$ in $(-a,a)\times\mathbb{R}^N$. Let us show that $\phi^c(s,x) < p(x)$.

Set

$$\gamma^* = \inf\left\{\gamma,\ \gamma p(x) > \phi^c(s,x) \text{ for all } (s,x)\ \in \overline{\Sigma_a}\right\}.$$

Since $p > 0$ and p is L-periodic with respect to x, there exists $\delta > 0$ such that $p > \delta$ in $\overline{\Sigma_a}$. Therefore γ^* does exist. Moreover since $\phi^c(a,x) = p(x)$, $\gamma^* \geq 1$. One has to show that $\gamma^* = 1$.

Assume $\gamma^* > 1$. By continuity, one has $\gamma^* p \geq \phi^c$ in $\overline{\Sigma_a}$. On the other hand, there exists a sequence $\gamma_n \to \gamma^*$, $\gamma_n < \gamma^*$ and a sequence (s_n, x_n) in $\overline{\Sigma_a}$ such that $\gamma_n p(x_n) \leq \phi^c(s_n,x_n)$. Since p and ϕ^c are L-periodic in x, one can assume that $x_n \in \overline{C}$. Up to the extraction of a subsequence, one can also assume that $(s_n,x_n) \to (s_1,x_1) \in [-a,a]\times\overline{C}$. Passing to the limit $n\to\infty$, one obtains $\gamma^* p(x_1) = \phi^c(s_1,x_1)$.

Next, set $z = \gamma^* p - \phi^c$. One has $f(x,\gamma^* p) \leq \gamma^* f(x,p)$ since $f(.,s)/s$ is supposed to be decreasing in s. Thus $L_\varepsilon(\gamma^* p) + f(x,\gamma^* p) \leq 0$. As a consequence, $L_\varepsilon z + f(x,\gamma^* p) - f(x,\phi^c) \leq 0$.

Hence,

$$\left\{\begin{array}{l} L_\varepsilon z + bz \leq 0 \\ z \geq 0 \end{array}\right. \text{ in } \Sigma_a, \tag{18}$$

where b is a bounded function (because f is globally lipschitz-continuous). Moreover, one has

$$\left\{\begin{array}{lcl} \gamma^* p(x) & > & \phi^c(-a,x) = 0, \\ \gamma^* p(x) & > & \phi^c(a,x) = p(x), \end{array}\right. \tag{19}$$

for all x in $\mathbb{R}^N$ (the last inequality follows from the assumption $\gamma^* > 1$ and from the positivity of p in $\mathbb{R}^N$). Therefore, the point (s_1,x_1) where z vanishes, lies in $(-a,a)\times\overline{C}$. Using (18) with the strong maximum principle, one obtains that $z \equiv 0$, which contradicts (19).

Thus $\gamma^* = 1$ and $\phi^c \leq p$. Using again the strong maximum principle, one obtains the inequality $\phi^c(s,x) < p(x)$ for all $(s,x) \in [-a,a)\times\mathbb{R}^N$.

In order to finish the proof of the lemma, we only have to follow the proof of Lemma 5.2 given in [13], which uses a sliding method in s (see [24]), replacing $\phi^c(a,x) = 1$ by $\phi^c(a,x) = p(x)$. □

Lemma 3 *The functions ϕ^c are decreasing and continuous with respect to c in the following sense : if $c > c'$, then $\phi^c < \phi^{c'}$ in Σ_a and if $c_n \to c \in \mathbb{R}$, then $\phi^{c_n} \to \phi^c$ in $C^2(\overline{\Sigma_a})$.*

PROOF. The proof is similar to that of lemma 5.3 in [13]. □

In the following, for any $\varepsilon > 0$, $a > 0$ and $c \in \mathbb{R}$, we call $\phi^c_{\varepsilon,a}$ the unique solution of (16) in $C^2\left(\overline{\Sigma_a}\right)$.

Set

$$p^- = \min_{x\in\overline{C}} p(x) = \min_{x\in\mathbb{R}^N} p(x) > 0 \text{ and } p^+ = \max_{x\in\overline{C}} = \max_{x\in\mathbb{R}^N} p(x) > 0.$$

Lemma 4 *There exist a_1 and K such that, for all $a \geq a_1$ and $\varepsilon \in (0,1]$,*

$$(c > K) \Rightarrow \left(\max_{x\in\overline{C}} \phi^c_{\varepsilon,a}(0,x) < \frac{p^-}{2}\right).$$

PROOF. Let $n \geq 2$ be an integer and g be a C^1 function defined on $[0, np^+]$, such that $g(0) = 0$, $g(np^+) = 0$, $g(u) > 0$ on $(0, np^+)$, $g'(np^+) < 0$. For n large enough, one can choose (using hypothesis (8)) g such that $f(x,u) \leq g(u)$ for all $x \in \mathbb{R}^N$ and $u \in [0, np^+]$.

Then, using a result of [21], one can assert that there exists c^1 such that the one-dimensional problem

$$\begin{cases} v'' - kv' + \dfrac{g(v)}{\alpha_0} = 0 \text{ in } \mathbb{R}, \\ v(-\infty) = 0 < v(.) < v(+\infty) = np^+ \text{ and } v(0) = \dfrac{p^-}{2}, \end{cases} \tag{20}$$

admits a unique solution v, for each $k \geq c^1 > 0$ (remember that $\alpha_0 > 0$ is given in (6)). Set $k = c^1$, and let $v = v(s)$ be the unique solution of (20) associated to $k = c^1$. It is also known that v is increasing in $\mathbb{R}$. Take $c > \max_{x\in\mathbb{R}^N} \{(eA(x)e + 1)k + \nabla\cdot(A(x)e)\}$, and $\varepsilon \in (0,1]$. One has

$$\begin{aligned} L_\varepsilon v + f(x,v) &= (eA(x)e + \varepsilon)v''(s) + (\nabla\cdot(A(x)e) - c)v'(s) + f(x, v(s)) \\ &= \{(eA(x)e + \varepsilon)k + \nabla\cdot(A(x)e) - c\}\, v'(s) \\ &\quad -(eA(x)e + \varepsilon)g(v(s))/\alpha_0 + f(x, v(s)), \end{aligned}$$

from (20). Thus

$$L_\varepsilon v + f(x,v) \leq \{(eA(x)e + \varepsilon)k + \nabla\cdot(A(x)e) - c\}\, v'(s) - \varepsilon g(v(s))/\alpha_0$$

since $eA(x)e \geq \alpha_0$ and $f(x,u) \leq g(u)$ for all $(x,u) \in \mathbb{R}^N \times [0, np^+]$. Moreover $g \geq 0$, $v' > 0$ and $(eA(x)e + \varepsilon)k + \nabla\cdot A(x)e - c < 0$, owing to the choice of c. As a consequence, one gets

$$L_\varepsilon v + f(x,v) < 0 \text{ in } \Sigma_a.$$

By using a sliding method as in Lemma 2, with v and $\phi^c_{\varepsilon,a}$, and by using the monotonicity of v and the fact that $v(-a) > 0$ and $v(a) > p^+ \geq p(x)$ for all $x \in \mathbb{R}^N$ and for a large enough, one can conclude that

$$\phi^c_{\varepsilon,a}(s,x) < v(s),$$

for all $(s,x) \in \overline{\Sigma_a}$ and for a large enough.

Hence, it follows that

$$\max_{x\in\mathbb{R}^N} \phi^c_{\varepsilon,a}(0,x) = \max_{x\in\overline{C}} \phi^c_{\varepsilon,a}(0,x) < v(0) = \frac{p^-}{2},$$

for $c > \max\limits_{\mathbb{R}^N} \{(eA(x)e+1)k + \nabla\cdot(A(x)e)\}$ and a large enough, which completes the proof of the lemma. □

Let us now consider the functions $\phi^0_{\varepsilon,a}$, associated to $c=0$. Take a sequence $a_n \to +\infty$. Let us pass to the limit $n \to +\infty$. From standard elliptic estimates and Sobolev's injections, the functions ϕ^0_{ε,a_n} converge (up to the extraction of a subsequence) in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, for all $0 \le \beta < 1$, to a function ϕ^0 which satisfies

$$\begin{cases} L_\varepsilon \phi^0 + f(x,\phi^0) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi^0 \text{ is L-periodic w.r.t. } x, \\ \phi^0 \text{ is nondecreasing w.r.t. } s, \end{cases}$$

with $c=0$. Furthermore, $0 \le \phi^0(s,x) \le p(x)$ for all $(s,x)\in\mathbb{R}\times\mathbb{R}^N$.

One then has the

Lemma 5 *There exist $x_1 \in \overline{C}$ and $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$, $\phi^0_{\varepsilon,a_n}(0,x_1) > \dfrac{p^-}{2}$.*

PROOF. Assume $\phi^0(0,x) < p(x)$ for all $x\in\overline{C}$. Then

$$0 \le \max_{x\in\overline{C}} \phi^0(0,x) < p^+.$$

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $(-a,a)$, such that

$$\max_{x\in\overline{C}} \phi^0_{\varepsilon,a_n}(s_n,x) = b := \frac{1}{2}p^+ + \frac{1}{2}\max_{x\in\overline{C}} \phi^0(0,x).$$

Let us show that $s_n > 0$ for n large enough.

Assume by contradiction that there exists a subsequence $a_k \to +\infty$ such that $s_k \le 0$ for all $k\in\mathbb{N}$. Then, since ϕ^0_{ε,a_k} is increasing in s,

$$b = \max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(s_k,x) \le \max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(0,x),$$

while

$$\max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(0,x) \to \max_{x\in\overline{C}} \phi^0(0,x) \text{ as } k\to+\infty.$$

Passing to the limit $k\to+\infty$, one obtains

$$b \le \max_{x\in\overline{C}} \phi^0(0,x).$$

That leads to a contradiction since $p^+ > \max\limits_{x\in\overline{C}} \phi^0(0,x)$. Therefore, one has shown that $s_n > 0$ for n large enough.

Set $\phi_{a_n}(s,x) = \phi^0_{\varepsilon,a_n}(s+s_n,x)$, defined on $(-a_n-s_n, a_n-s_n)\times\mathbb{R}^N$. Then

$$\max_{x\in\overline{C}} \phi_{a_n}(0,x) = b.$$

One easily sees that $-a_n - s_n \to -\infty$ as $n \to +\infty$. Then two cases may occur, up to the extraction of some subsequence :

case 1 : $a_n - s_n \to +\infty$. From standard elliptic estimates and Sobolev's injections, the functions ϕ_{a_n} converge (up to the extraction of a subsequence) in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, for all $0 \leq \beta < 1$, to a nonnegative function ϕ satisfying

$$\left\{\begin{array}{rcl} L_\varepsilon\phi + f(x,\phi) & = & 0 \text{ in } \mathbb{R}\times\mathbb{R}^N \text{ (with } c=0), \\ \phi \text{ is L-periodic w.r.t. } x, & & \\ \phi \text{ is nondecreasing w.r.t. } s, & & \\ 0 \leq \phi & \leq & p(x) \\ \max_{x\in\overline{C}} \phi(0,x) & = & b. \end{array}\right. \tag{21}$$

Moreover, from standard elliptic estimates, from the monotonicity of ϕ, and from the periodicity in x, it follows that

$$\phi(s,x) \to \phi_\pm(x) \text{ in } C^2(\mathbb{R}^N) \text{ as } s \to \pm\infty,$$

where each function $\phi_\pm$ satisfies

$$\left\{\begin{array}{l} \nabla\cdot(A(x)\nabla\phi_\pm) + f(x,\phi_\pm) = 0 \text{ in } \mathbb{R}^N, \\ \phi_\pm \text{ is L-periodic}, \\ \phi_\pm \geq 0. \end{array}\right.$$

From the uniqueness Theorem 2.1 of [17], one can conclude that either $\phi_\pm(x) \equiv 0$ or $\phi_\pm(x) \equiv p(x)$. Moreover, $\phi_1^-(x) \leq \phi_1(0,x)$ because of the monotonicity of ϕ_1 with respect to s. Therefore $\phi_1^-(x) \leq b < p^+$. Thus, $\phi_1^-(x) \not\equiv p(x)$, and $\phi_1^- \equiv 0$. Similarly, $\phi_1^+(x) \geq \phi_1(0,x)$. Thus there exists $x_0 \in \overline{C}$ such that $\phi_1^+(x_0) \geq b > 0$. As a consequence, $\phi_1^+ \equiv p$.

Next, multiply the equation (21) (with $c = 0$) by ϕ_s and integrate it over $(-N,N)\times C$, where N is a positive real number. Then,

$$\begin{aligned} \int_{(-N,N)\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s ds dx + \int_{(-N,N)\times C} \nabla_x\cdot(A(x)\nabla_x\phi)\phi_s ds dx \\ + \int_{(-N,N)\times C} \{\nabla_x\cdot(A(x)e\phi_s) + \partial_s(eA(x)\nabla_x\phi)\}\,\phi_s ds dx \\ + \int_{(-N,N)\times C} f(x,\phi)\phi_s ds dx &= 0. \end{aligned} \tag{22}$$

First, one has

$$\int_{(-N,N)\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s ds dx = \frac{1}{2}\int_C \left[(eA(x)e+\varepsilon)(\phi_s)^2\right]_{-N}^{N} ds dx. \tag{23}$$

From standard elliptic estimates, one knows that $\phi_s \to 0$ as $s \to +\infty$. Passing to the limit $N \to +\infty$ in (23), one obtains

$$\int_{\mathbb{R}\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s ds dx = 0. \tag{24}$$

Next, using an integration by parts over $(-N,N)\times C$ and the periodicity of ϕ with respect to x, one has

$$\begin{aligned}\int_{(-N,N)\times C}\nabla_x\cdot(A(x)\nabla_x\phi)\phi_s dsdx &= -\int_{(-N,N)\times C}\nabla_x\phi_s\cdot A(x)\nabla_x\phi dsdx\\ &= -\frac{1}{2}\int_{(-N,N)\times C}(\nabla_x\phi\cdot A(x)\nabla_x\phi)_s dsdx\\ &= -\frac{1}{2}\int_C\left[\nabla_x\phi\cdot A(x)\nabla_x\phi\right]_{-N}^{N}dsdx,\end{aligned}\tag{25}$$

since the matrix field $A(x)$ is symmetric. Passing to the limit $N\to+\infty$ in (25), one obtains, using standard elliptic estimates :

$$\int_{\mathbb{R}\times C}\nabla_x\cdot(A(x)\nabla_x\phi)\phi_s dsdx = -\frac{1}{2}\int_C\nabla p\cdot(A(x)\nabla p)dsdx.\tag{26}$$

Next, from the periodicity of ϕ with respect to x, one can similarly show that

$$\int_{\mathbb{R}\times C}\left\{\nabla_x\cdot(A(x)e\phi_s)+\partial_s(eA(x)\nabla_x\phi)\right\}\phi_s dsdx = 0.\tag{27}$$

Set $F(x,u)=\displaystyle\int_0^u f(x,s)ds$. Then,

$$\int_{\mathbb{R}\times C}f(x,\phi)\phi_s dsdx = \int_{\mathbb{R}\times C}F(x,\phi(s,x))_s dsdx = \int_C F(x,p(x))dx.\tag{28}$$

Passing to the limit $N\to+\infty$ in (22), and using (24), (26), (27) and (28), one gets

$$\int_C\left[F(x,p(x))-\frac{1}{2}\nabla p\cdot(A(x)\nabla p)\right]dx = 0.\tag{29}$$

Moreover, using a property on the energy of p, which has been established in Proposition 3.7 of [17], one asserts that

$$\int_C\left[F(x,p(x))-\frac{1}{2}\nabla p\cdot(A(x)\nabla p)\right]dx =: -E(p) > 0.$$

The latter is in contradiction with (29), therefore case 1 is ruled out.

case 2 : $a_n - s_n \to d < +\infty$. Up to the extraction of some subsequence, the functions ϕ_{a_n} converge in $C^{2,\beta}_{loc}((-\infty,d)\times\mathbb{R}^N)$ (for all $0\le\beta<1$) to a function ϕ satisfying (21), with $c=0$, in the set $(-\infty,d)\times\mathbb{R}^N$. Moreover, the family of functions (ϕ_{a_n}) is equi-Lipschitz-continuous in any set of the type $[a_n-s_n-1,a_n-s_n]\times\overline{C}$. Therefore, for all $\eta>0$, there exists $\kappa>0$ such that

$$\forall x\in\overline{C},\ \forall n,\ \forall s\in[a_n-s_n-\kappa,a_n-s_n],\ p(x)-\eta\le\phi_{a_n}(s,x)\le p(x).\tag{30}$$

Then choose $x_0\in\overline{C}$ such that $p(x_0)=p^+$. Formula (30) applied on x_0 together with

$$\max_{x\in\overline{C}}\phi_{a_n}(0,x)=b<p^+$$

implies that $a_n-s_n>\delta$ for some $\delta>0$. Hence $d>0$ and

$$\max_{x\in\overline{C}}\phi(0,x)=b.$$

Moreover (30) implies that ϕ can be extended by continuity on $\{d\} \times \mathbb{R}^N$ with $\phi(d,x) = p(x)$. Furthermore, from standard elliptic estimates up to the boundary, the function ϕ is actually in $C^1\left((-\infty, d] \times \mathbb{R}^N\right)$.

Following the proof of case 1, one shows that $\phi(-\infty,.) \equiv 0$. The next steps are similar to those of case 1. One gets a contradiction.

Therefore, there exists $x_1 \in \overline{C}$ such that $\phi^0(0,x_1) = p(x_1)$. Hence, taking any sequence $a_n \to +\infty$ such that the sequence $(\phi^0_{\varepsilon,a_n})$ converges in $C^{2,\beta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$ for all $0 \leq \beta < 1$, there exists N_0 such that for all $n \geq N_0$, $\phi^0_{\varepsilon,a_n}(0,x_1) > \dfrac{p^-}{2}$. That completes the proof of Lemma 5. □

Finally, one gets

Proposition 1 *Fix $\varepsilon \in (0,1]$. Let $a_n \to +\infty$ be the sequence defined above. Then, there exist $K \in \mathbb{R}$, $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ there exists a unique real number $c = c^{\varepsilon,a_n}$ such that ϕ^c_{ε,a_n} satisfies the normalization condition*

$$\max_{x\in\mathbb{R}^N} \phi^c_{\varepsilon,a_n}(0,x) = \max_{x\in\overline{C}} \phi^c_{\varepsilon,a_n}(0,x) = \frac{p^-}{2}. \tag{31}$$

Furthermore,

$$\forall\ 0 < \varepsilon \leq 1,\ \forall\ n \geq N_1,\ 0 < c^{\varepsilon,a_n} < K.$$

PROOF. Fix $\varepsilon \in (0,1]$. Under the notations of the two preceding lemmas, let us define N_1 such that $a_{N_1} > a_1$ and $N_1 > N_0$. It follows from this lemmas that for each $n \geq N_1$,

$$\begin{cases} \forall c \geq K,\ \max\limits_{x\in\mathbb{R}^N} \phi^c_{\varepsilon,a_n}(0,x) < \dfrac{p^-}{2}, \\ \text{for } c = 0,\ \max\limits_{x\in\mathbb{R}^N} \phi^0_{\varepsilon,a_n}(0,x) > \dfrac{p^-}{2}. \end{cases}$$

On the other hand, Lemma 3 yields that, for each $n \geq N_1$, the function

$$\Xi(c) = \max_{x\in\mathbb{R}} \phi^c_{\varepsilon,a_n}(0,x)$$

is decreasing and continuous with respect to c. Therefore the proposition follows. □

2.2 Passage to the limit in the infinite domains

Using the result of Proposition 1, we are going to pass to the limit $n \to +\infty$ in the infinite cylinder $\mathbb{R} \times \mathbb{R}^N$ for the solutions $\phi^{c^{\varepsilon,a_n}}_{\varepsilon,a_n}$ satisfying (31).

Proposition 2 *Under the notations of Proposition 1, one has*

$$\forall \varepsilon > 0,\ 0 < c^\varepsilon := \liminf_{n\to+\infty, n\geq N_1} c^{\varepsilon,a_n} \leq K.$$

PROOF. From Proposition 1, one has $0 \le c^\varepsilon \le K$. Up to the extraction of a subsequence, one can assume $c^{\varepsilon,a_n} \to c^\varepsilon$ as $n \to +\infty$ and $\phi_{\varepsilon,a_n}^{c^{\varepsilon,a_n}} \to \phi$ in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, for all $0 \le \beta < 1$, where ϕ satisfies

$$\begin{cases} L_\varepsilon \phi + f(x,\phi) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi \text{ is L-periodic w.r.t. } x, \\ \phi \text{ is nondecreasing w.r.t. } s, \end{cases}$$

with $c = c^\varepsilon$ and

$$\max_{x\in\mathbb{R}^N} \phi(0,x) = \frac{p^-}{2}.$$

Then, following the calculus of Lemma 5, case 1, one can assert that

$$\phi(-\infty,x) = 0,\ \ \phi(+\infty,x) = p(x) \text{ for all } x \in \mathbb{R}, \text{ and}$$

$$c^\varepsilon \int_{\mathbb{R}\times C} (\phi_s)^2 ds dx = \int_C \left[F(x,p) - \frac{1}{2}\nabla p \cdot (A(x)\nabla p) \right] dx = -E(p) > 0, \tag{32}$$

from Proposition 3.7 of [17]. Therefore $c^\varepsilon > 0$. □

Proposition 3 *Up to the extraction of some subsequence, $\phi_{\varepsilon,a_n}^{c^{\varepsilon,a_n}}$ converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \le \beta < 1$), to a function ϕ^ε such that, in $\mathbb{R}\times\mathbb{R}^N$,*

$$\begin{cases} \nabla_x \cdot (A(x)\nabla_x\phi^\varepsilon) + (eA(x)e + \varepsilon)\phi^\varepsilon_{ss} + \nabla_x \cdot (A(x)e\phi^\varepsilon_s) & \\ \qquad + \partial_s(eA(x)\nabla_x\phi^\varepsilon) - c\phi^\varepsilon_s + f(x,\phi^\varepsilon) & = 0, \\ \phi^\varepsilon \text{ is L-periodic w.r.t. } x, & \\ \max_{x\in\mathbb{R}^N} \phi^\varepsilon(0,x) & = \frac{p^-}{2}, \\ \phi^\varepsilon \text{ is increasing w.r.t. } s. & \end{cases}$$

Furthermore, $\phi^\varepsilon(-\infty,x) = 0$ and $\phi^\varepsilon(+\infty,x) = p(x)$ for all $x \in \mathbb{R}^N$.

PROOF. The convergence follows from the same arguments that were invoqued in the preceding propositions. Moreover, ϕ^ε in nondecreasing w.r.t. s because each $\phi_{\varepsilon,a_n}^{c^{\varepsilon,a_n}}$ is increasing in s. The limits $\phi^\varepsilon(-\infty,x) = 0$ and $\phi^\varepsilon(+\infty,x) = p(x)$ can be proved in the same way as in Lemma 5, case 1, using

$$\max_{x\in\mathbb{R}^N} \phi^\varepsilon(0,x) = \frac{p^-}{2},$$

and the fact that ϕ^ε in nondecreasing in s. The only thing that it remains to prove is that ϕ^ε is increasing in s.

For any $h > 0$, the function $\phi^\varepsilon(s+h,x) - \phi^\varepsilon(s,x)$ is a nonnegative and nonconstant solution of a linear elliptic equation with bounded coefficients. It follows then from the strong maximum principle that $\phi^\varepsilon(s+h,x) - \phi^\varepsilon(s,x) > 0$, for all $(s,x) \in \mathbb{R}\times\mathbb{R}^N$. That proves that the function ϕ^ε is increasing in the variable s. □

2.3 Passage to the limit $\varepsilon \to 0$

Our first aim is to prove that the real numbers c^ε are bounded from below by a positive constant.

Proposition 4 *Under the notations of Proposition 1, one has*

$$0 < \liminf_{\varepsilon \to 0} c^\varepsilon \leq K.$$

PROOF. From Proposition 1, for each $\varepsilon > 0$, one has $0 < c^\varepsilon \leq K$. Assume that there exists a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$ such that $c^{\varepsilon_n} \to 0$ as $n \to +\infty$. In the sequel, for the sake of simplicity, we drop the index n. Set $u^\varepsilon(t,x) = \phi^\varepsilon(x \cdot e + c^\varepsilon t, x)$. Then u^ε is a classical solution of

$$\begin{cases} \dfrac{\varepsilon}{(c^\varepsilon)^2} u^\varepsilon_{tt} + \nabla_x \cdot (A(x)\nabla_x u^\varepsilon) - u^\varepsilon_t + f(x,u^\varepsilon) = \ 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ \forall\, k \in \displaystyle\prod_{i=1}^N L_i\mathbb{Z},\ u^\varepsilon\left(t + \dfrac{k \cdot e}{c}, x\right) = u^\varepsilon(t, x+k) \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u^\varepsilon(t,x) \to 0 \text{ as } t \to -\infty,\ u^\varepsilon(t,x) \to p(x) \text{ as } \quad t \to +\infty. \end{cases} \tag{33}$$

Moreover, $0 < u^\varepsilon(t,x) < p(x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Lastly, since ϕ is increasing in the variable s and $c^\varepsilon > 0$, each function u^ε is increasing in the variable t.

Up to the extraction of some subsequence, as it was said in [13] (Proposition 5.10), three cases may occur :

$$\frac{\varepsilon}{(c^\varepsilon)^2} \to \kappa \in (0,+\infty),\ \frac{\varepsilon}{(c^\varepsilon)^2} \to +\infty \text{ or } \frac{\varepsilon}{(c^\varepsilon)^2} \to 0 \ \text{ as } \varepsilon \to 0^+.$$

Let us study the :

case 1 : Assume $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to \kappa \in (0,+\infty)$. Let x_0 $\mathbb{R}^N$ be such that $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$ and $x_0 \cdot e > 0$. Since $u^\varepsilon(t,x_0) \to 0$ as $t \to -\infty$ and $u^\varepsilon(t,x_0) \to p(x_0)$ as $t \to +\infty$ from our assumptions on u^ε, one can assume, up to translation with respect to t, that $u^\varepsilon(0,x_0) = \dfrac{p^-}{2}$.

Since $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to \kappa$, standard elliptic estimates imply that the functions u^ε converge (up to extraction of some subsequence) in $C^{2,\beta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$ (for all $0 \leq \beta < 1$) to a function u satisfying

$$\begin{cases} \kappa u_{tt} + \nabla_x \cdot (A(x)\nabla_x u) - u_t + f(x,u) \ = \ 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ 0 \leq u \leq p,\ u_t \ \geq \ 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \end{cases}$$

and $u(0,x_0) = \dfrac{p^-}{2}$. Now, fix any $B \in \mathbb{R}$. Since $c^\varepsilon \to 0^+$ and $x_0 \cdot e > 0$, $B < \frac{x_0 \cdot e}{c^\varepsilon}$ for ε sufficiently small. Thus, as u^ε is increasing in t, one has $u^\varepsilon(B,0) \leq u^\varepsilon(\frac{x_0 \cdot e}{c^\varepsilon}, 0)$. But $u^\varepsilon(\frac{x_0 \cdot e}{c^\varepsilon}, 0) = u^\varepsilon(0,x_0) = \dfrac{p^-}{2}$. Therefore, passing to the limit $\varepsilon \to 0$, one obtains

$$\forall\, B > 0, \quad u(B,0) \leq \frac{p^-}{2}. \tag{34}$$

Let u^+ be the function defined in $\mathbb{R}^N$ by $u^+(x) = \lim_{t \to +\infty} u(t,x)$. This function can be defined since u is bounded and nondecreasing in t. From standard elliptic estimates, the convergence holds in $C^2_{loc}(\mathbb{R}^N)$, and u^+ solves

$$\nabla \cdot (A(x)\nabla u^+) + f(x,u^+) = 0 \text{ in } \mathbb{R}^N \tag{35}$$

with $0 \leq u^+(x) \leq p(x)$ for all $x \in \mathbb{R}^N$. But it follows from our hypotheses on f and from Theorems 2.1 and 2.3 of [17] that the equation (35) admits exactly two nonnegative solutions, which are 0 and p. Therefore, as $u(0,x_0) = \frac{p^-}{2}$ and $u_t \geq 0$, one has $u^+(x_0) \geq \frac{p^-}{2} > 0$, thus $u^+ \equiv p$. However, (34) gives $u^+(0) \leq \frac{p^-}{2}$. As a consequence, u^+ cannot be equal to p and case 1 is ruled out.

case 2 : Assume that $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to +\infty$. As it was done in [13] (Proposition 5.10), one makes the change of variables $\tau = (c^\varepsilon/\sqrt{\varepsilon})t$. The function $v^\varepsilon(\tau,x) = u^\varepsilon\left(\dfrac{\sqrt{\varepsilon}}{c^\varepsilon}\tau, x\right)$ satisfies

$$\begin{cases} v^\varepsilon_{\tau\tau} + \nabla_x\cdot(A(x)\nabla_x v^\varepsilon) - \dfrac{c^\varepsilon}{\sqrt{\varepsilon}}v^\varepsilon_\tau + f(x,v^\varepsilon) = \quad 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \forall\, k \in \displaystyle\prod_{i=1}^N L_i\mathbb{Z}\ v^\varepsilon\left(\tau + \frac{k\cdot e}{\sqrt{\varepsilon}}, x\right) = \quad v^\varepsilon(\tau, x+k) \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ v^\varepsilon(\tau,x)\to 0 \text{ as } \tau\to-\infty,\ v^\varepsilon(\tau,x)-p(x)\to 0 \quad \text{as } \tau\to+\infty. \end{cases}$$

Moreover, $0 < v^\varepsilon < p$ and v^ε is nondecreasing with respect to τ. Furthermore, as it was done in case 1, one can assume that $v^\varepsilon(0,x_0) = \frac{p^-}{2}$ for some $x_0 \in \mathbb{R}^N$ such that $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$ and $x_0\cdot e > 0$. Since $c^\varepsilon/\sqrt{\varepsilon} \to 0^+$, the functions v^ε converge (up to the extraction of some subsequence) in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \leq \beta < 1$) to a function v which satisfies

$$\begin{cases} v_{\tau\tau} + \nabla_x\cdot(A(x)\nabla_x v) + f(x,v) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ 0 \leq v \leq p,\ v_\tau \geq 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \end{cases}$$

and $v(0,x_0) = \frac{p^-}{2}$. Moreover, as in case 1, one can show that

$$\forall\, B > 0, \quad v(B,0) \leq \frac{p^-}{2}.$$

By defining $v^+(x) := \lim_{\tau\to+\infty} v(\tau,x)$, one can also obtain a contradiction the same way as in case 1.

case 3 : Assume that $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to 0$. The elliptic operators in (33) become degenerate at the limit $\varepsilon \to 0$, and one cannot use the same arguments as in cases 1 and 2.

In order to pass to the limit $\varepsilon \to 0$, let us state two new inequalities on u^ε.

Using the same calculations as those which were used to prove (29) and (32), and making the change of variables $t = \dfrac{s - x\cdot e}{c^\varepsilon}$, one obtains

$$\forall\, \varepsilon \in (0,1), \int_{\mathbb{R}\times C} (u^\varepsilon_t)^2 = -E(p).$$

Therefore, the periodicity condition in (33) gives us that

$$\forall \varepsilon \in (0,1),\ \forall n \in \mathbb{N}, \quad \int_{\mathbb{R}\times(-nL_1,nL_1)\times\cdots\times(-nL_N,nL_N)} (u^\varepsilon_t)^2 = -(2n)^N E(p). \tag{36}$$

Similarly, multiplying equation (33) by 1 and u^ε, one gets the existence of $\gamma \geq 0$ such that

$$\forall \varepsilon \in (0,1),\ \forall n \in \mathbb{N}, \quad \int_{\mathbb{R}\times(-nL_1,nL_1)\times\cdots\times(-nL_N,nL_N)} f(x,u^\varepsilon) + |\nabla_x u^\varepsilon|^2 \leq (2n)^N\gamma. \tag{37}$$

Next, using Theorem A.1 of [13] (see also [14]), one has the following a priori estimate :

Lemma 6 *There exists a constant M, which does not depend on ε, such that the function u^ε solving (33) satisfies*

$$|\nabla_x u^\varepsilon| \leq M \text{ in } \mathbb{R}\times\mathbb{R}^N \tag{38}$$

for ε small enough.

The above estimates were proved in [14] with a Bernstein-type method. This method uses the maximum principle applied to some quantities involving $|\nabla_x u^\varepsilon|^2$, and it requires that the functions u^ε are of class C^3. The latter is true here because of the smoothness assumptions on A and f.

From (36) and (38), and arguing as in [13] (Proposition 5.10), we obtain that u^ε converges (up to the extraction of some subsequence) almost everywhere in $\mathbb{R}\times\mathbb{R}^N$ to a function $u \in H^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$ and

$$(u^\varepsilon, u^\varepsilon_t, \nabla_x u^\varepsilon) \rightharpoonup (u, u_t, \nabla_x u) \text{ in } L^2(\mathcal{K}),$$

for every compact subset $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}^N$. From (36) and (38), one can actually assume that $u^\varepsilon \to u$ in $L^2_{loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ strong. Moreover, $0 \leq u \leq p(x)$, $u_t \geq 0$ and from (36), (37) and (38),

$$\int_{\mathbb{R}\times\mathcal{K}_1} |\nabla_x u|^2 + (u_t)^2 \leq C(\mathcal{K}_1), \tag{39}$$

for every compact subset $\mathcal{K}_1 \subset \mathbb{R}^N$.

From parabolic regularity, u is then a classical solution of

$$\left\{\begin{array}{rcl} u_t - \nabla_x\cdot(A(x)\nabla_x u) - f(x,u) & = & 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ 0\leq u\leq p,\ u_t & \geq & 0 \text{ in } \mathbb{R}\times\mathbb{R}^N. \end{array}\right. \tag{40}$$

Moreover, one can assume, up to a translation in t, that

$$\forall\ \varepsilon>0,\ \int_{(0,1)\times C} u^\varepsilon(t, x+x_0)dxdt = |C|\frac{p^-}{2}. \tag{41}$$

for some $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$ such that $x_0\cdot e>0$. Since $c^\varepsilon \to 0^+$ and u^ε is increasing in t, it follows that for all $B\in\mathbb{R}$, and for ε sufficiently small,

$$\forall\ (t,x)\in(0,1)\times C,\ u^\varepsilon(B+t,x) \leq u^\varepsilon(t+\frac{x_0\cdot e}{c^\varepsilon}, x) = u^\varepsilon(t, x+x_0)$$

from (33). Next, one integrates over $(0,1)\times C$ and passes to the limit $\varepsilon\to 0^+$. By using (41) and the fact that $u^\varepsilon \rightharpoonup u$ in $L^2_{loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ weak,[7] one obtains

$$\forall\ B\in\mathbb{R},\ \int_{(0,1)\times C} u(B+t,x)dxdt \leq |C|\frac{p^-}{2}. \tag{42}$$

Using the monotonicity of u in t, let us define $u^+(x) = \lim_{t\to+\infty} u(t,x)$. Then, one has $u^+\geq 0$ and $\nabla_x\cdot(A(x)\nabla_x u^+) + f(x,u^+) = 0$. Therefore, as it was stated in Theorems 2.1 and 2.3 of [17], $u^+\equiv 0$ or $u^+\equiv p$. But (41), passing to the limit $\varepsilon\to 0$ and $t\to+\infty$, rules out the case $u^+\equiv 0$. Hence $u^+\equiv p$. Next, using (42), and since u is nonincreasing in t, one obtains

$$\int_{(0,1)\times C} p(x)dx \leq |C|\frac{p^-}{2}$$

which is impossible. The proof of Proposition 4 is complete. □

[7] This convergence is actually strong.

2.4 Existence of a solution (c^1, u^1)

Let us choose a subsequence $\varepsilon \to 0$ such that $c^\varepsilon \to c^1 > 0$. For each ε, set $u^\varepsilon(t,x) = \phi^\varepsilon(x \cdot e + ct, x)$. As it was done in case 3 of Proposition 4, the functions u^ε converge (up to the extraction of some subsequence), in $H^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$ weak, and almost everywhere, to a classical solution u^1 of (40).

Let us prove that $u^1\left(t + \frac{k \cdot e}{c^1}, x\right) = u^1(t, x+k)$ for all $k \in \prod_{i=1}^N L_i\mathbb{Z}$ and for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. For all $B > 0$ and every compact set $\mathcal{K}_1$ in $\mathbb{R}^N$, one has,

$$\int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon(t, x+k)\right]^2 = \int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon\left(t + \frac{k\cdot e}{c^\varepsilon}, x\right)\right]^2,$$

whence

$$\int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon(t, x+k)\right]^2 \leq \left(\frac{k\cdot e}{c^1} - \frac{k\cdot e}{c^\varepsilon}\right)^2 \int_{\mathbb{R}\times\mathcal{K}_1} (u^\varepsilon_t)^2 \leq \left(\frac{k\cdot e}{c^1} - \frac{k\cdot e}{c^\varepsilon}\right)^2 C(\mathcal{K}_1)$$

from (39). Therefore, by passing to the limit $\varepsilon \to 0$, we obtain

$$u^1\left(t + \frac{k\cdot e}{c^1}, x\right) = u^1(t, x+k) \tag{43}$$

almost everywhere in $\mathbb{R} \times \mathbb{R}^N$. Since u^1 is continuous, the equality holds for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

In order to obtain our result, one has to prove that $u^1(t,x) \to 0$ as $x \cdot e \to -\infty$ and $u^1(t,x) - p(x) \to 0$ as $x \cdot e \to +\infty$, locally in t. Since u^1 verifies (43), and $c > 0$, it is equivalent to prove that $u^1(t,x) \to 0$ as $t \to -\infty$ and $u^1(t,x) \to p(x)$ as $t \to +\infty$, locally in x.

As it was done in the proof of Proposition 4, case 3, one can assume, up to a translation in t, that

$$\forall\ \varepsilon, \int_{(0,1)\times C} u^\varepsilon(t,x)dxdt = |C|\frac{p^-}{2}. \tag{44}$$

Since u^1 is bounded and nondecreasing with respect to t, one can define

$$u^\pm(t,x) = \lim_{t\to\pm\infty} u^1(t,x).$$

As done above, one knows that $u^\pm$ satisfies $\nabla\cdot(A(x)\nabla u^\pm) + f(x, u^\pm) = 0$ and one has $0 \leq u^\pm \leq p$. As already said, this equation admits exactly two nonnegative solutions, which are not larger than p, namely 0 and p.

Passing to the limit $\varepsilon \to 0$ in (44), and using the fact that u^1 is nondecreasing with respect to t, one has

$$\int_C u^+(x) \geq |C|\frac{p^-}{2}, \tag{45}$$

and

$$\int_C u^-(x) \leq |C|\frac{p^-}{2}. \tag{46}$$

One then easily concludes from (45) that u^+ is not equal to 0, and therefore $u^+ \equiv p$, and from (46), u^- is not equal to p, thus $u^- \equiv 0$.

From strong parabolic maximum principle, one obtains that u^1 is increasing in t. The existence result follows.

2.5 Existence of a solution (c, u) for all $c > c^1$

Proposition 5 *For each $c > c^1$, there exists a solution u of (10-11), associated to the speed c, and u is increasing in t.*

PROOF. Set $\phi^1(s,x) = u^1\left(\dfrac{s - x \cdot e}{c}, x\right)$, and, as before, define L_ε by

$$L_\varepsilon \phi = \nabla_x \cdot (A(x)\nabla_x \phi) + (eA(x)e + \varepsilon)\phi_{ss} + \nabla_x \cdot (A(x)e\phi_s) + \partial_s(eA(x)\nabla_x \phi) - c\phi_s.$$

Then, as it was done in [13] (Proposition 6.3), using Krylov-Safonov-Harnack type inequalities applied to $v = \partial_t u^1$, one gets the existence of a constant C such that $|\partial_{tt} u^1| \le C\partial_t u^1$ in $\mathbb{R} \times \mathbb{R}^N$, whence

$$L_\varepsilon \phi^1 + f(x, \phi^1) = \varepsilon \phi^1_{ss} + (c^1 - c)\phi^1_s < 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \tag{47}$$

for $\varepsilon > 0$ small enough. In what follows, let $\varepsilon > 0$ be small enough so that (47) holds.

For any $a \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, set

$$h_\tau = \min_{\overline{C}} \phi^1(-a + \tau, .).$$

With a similar method as in Lemma 1, one can show the existence of a solution $\phi_\tau \in C^2\left(\overline{\Sigma_a}\right)$ of the following problem :

$$\left\{\begin{array}{rcl} L_\varepsilon \phi_\tau + f(x, \phi_\tau) & = & 0 \text{ in } \Sigma_a, \\ \phi_\tau \text{ is L-periodic w.r.t. } x, & & \\ \phi_\tau(-a, x) = h_\tau \dfrac{p(x)}{p^+},\ \phi_\tau(a, x) & = & \phi^1(a + \tau, x) \text{ for all } x \in \mathbb{R}^N. \end{array}\right. \tag{48}$$

Let us now show that $h_\tau \dfrac{p(x)}{p^+} < \phi_\tau$ for all $(s, x) \in \Sigma_a$. First, since $f(x, u) = 0$ for all $u \le 0$, one has, from the strong maximum principle, $\phi_\tau > 0$ in $\overline{\Sigma_a}$. Therefore, one can define

$$\gamma^* = \sup\left\{\gamma > 0,\ \phi_\tau > \gamma h_\tau \frac{p(x)}{p^+} \text{ in } \Sigma_a\right\}.$$

Assume that $\gamma^* < 1$. As in the proof of Lemma 2, using the fact that $\gamma^* h_\tau p(x)/p^+ < \phi_\tau(\pm a, x)$ for all $x \in \mathbb{R}^N$ (since $p(x)/p^+ \le 1$ and ϕ^1 is increasing w.r.t. s), one gets the existence of $(s^*, x^*) \in (-a, a) \times C$ such that $\gamma^* h_\tau p(x)/p^+ \le \phi_\tau(s, x)$ for all $(s, x) \in [-a, a] \times \mathbb{R}^N$, with equality at $(s^*, x^*) \in (-a, a) \times \mathbb{R}^N$. On the other hand,

$$L_\varepsilon(h_\tau \gamma^* \frac{p}{p^+}) = \gamma^* \frac{h_\tau}{p^+} L_\varepsilon(p) > -f(x, \gamma^* h_\tau \frac{p}{p^+}),$$

since $\gamma^* h_\tau / p^+ < 1$, and since $f(., s)/s$ is decreasing in s from our hypothesis on f. That leads to a contradiction as in Lemma 2.

Therefore, $\gamma^* \ge 1$, whence $\phi_\tau \ge h_\tau p/p^+$, and the strong maximum principle yields

$$\forall\ (s, x) \in \Sigma_a,\ h_\tau \frac{p(x)}{p^+} < \phi_\tau(s, x). \tag{49}$$

Similarly, one can easily show that $\phi_\tau(s, x) < p(x)$ for all $(s, x) \in \overline{\Sigma_a}$. Therefore, $\phi^1(s + \tau + k, x) \ge \phi_\tau(s, x)$ in $\overline{\Sigma_a}$ for k large enough. Let $\overline{k}$ be the smallest k such that the latter holds.

From the boundary conditions in (48), one knows that $\overline{k} \geq 0$. Assume $\overline{k} > 0$. By continuity, it necessarily follows that $\phi^1(s+\tau+\overline{k}, x) \geq \phi_\tau(s,x)$ with equality at a point $(\overline{s}, \overline{x}) \in \overline{\Sigma_a}$. Since ϕ^1 is increasing in s,

$$\phi^1(-a+\tau+\overline{k},\cdot) > \phi^1(-a+\tau,\cdot) \geq h_\tau \geq h_\tau \frac{p(\cdot)}{p^+} = \phi^\tau(-a,\cdot),$$

and $\phi^1(a+\tau+\overline{k},\cdot) > \phi^1(a+\tau,\cdot) = \phi^\tau(a,\cdot)$. Therefore $(\overline{s},\overline{x}) \in (-a,a)\times\overline{C}$ (one can assume this using the L-periodicity in x of ϕ^1 and ϕ_τ). But, from (47), it is found that $\phi^1(s+\tau+\overline{k},x)$ is a supersolution of (48). Therefore, the strong maximum principle implies that $\phi^1(s+\tau+\overline{k},x) = \phi_\tau(s,x)$ in $\overline{\Sigma_a}$. One gets a contradiction with the boundary condition at $s = a$. As a consequence, $\overline{k} = 0$ and one has

$$\forall\ (s,x) \in \overline{\Sigma_a},\ \phi_\tau(s,x) \leq \phi^1(s+\tau,x). \tag{50}$$

Since ϕ^1 is increasing in s, it also follows that $\phi_\tau(s,x) < \phi^1(a+\tau,x)$ in Σ_a.

As a conclusion, from (49) and (50), one has

$$\forall (s,x)\ \in\ \Sigma_a,\ h_\tau \frac{p(x)}{p^+} < \phi_\tau(s,x) < \phi^1(a+\tau,x).$$

Using the same sliding method as in Lemma 5.2 in [13], it follows that ϕ_τ is increasing in s and is the unique solution of (48) in $C^2\left(\overline{\Sigma_a}\right)$. Moreover, using the fact that the boundary conditions for ϕ_τ at $s = \pm a$ are increasing in τ, one can prove, as in Lemma 5.3 in [13], that the functions ϕ_τ are continuous with respect to τ in $C^2\left(\overline{\Sigma_a}\right)$ and increasing in τ. But, since $\phi^1(-\infty,x) = 0$ and $\phi^1(+\infty,x) = p(x)$ in $\mathbb{R}^N$, it follows from (49) and (50) that $\phi_\tau \to 0$ as $\tau \to -\infty$ uniformly in $\overline{\Sigma_a}$ and that,

$$\forall\ \alpha > 0,\ \exists\ T,\ \forall\ \tau > T, \quad \phi_\tau(s,x) > \frac{p^-}{p^+}p - \alpha \text{ in } \overline{\Sigma_a}.$$

Therefore, for each $a > 1$, there exists a unique $\tau_\varepsilon(a) \in \mathbb{R}$ such that $\phi^{\varepsilon,a} := \phi_{\tau_\varepsilon(a)}$ solves (48) and satisfies

$$\int_{(0,1)\times C} \phi^{\varepsilon,a}(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C|\min(c,1). \tag{51}$$

Let $a_n \to +\infty$. From standard elliptic estimates, the functions ϕ^{ε,a_n} converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \leq \beta < 1$), up to the extraction of a subsequence, to a function ϕ^ε satisfying

$$\left\{\begin{array}{rcl} L_\varepsilon\phi^\varepsilon + f(x,\phi^\varepsilon) & = & 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi^\varepsilon \text{ is L-periodic w.r.t. } x. & & \end{array}\right. \tag{52}$$

Moreover, ϕ^ε is nonincreasing with respect to s, satisfies $0 \leq \phi^\varepsilon(s,x) \leq p(x)$ in $\mathbb{R}\times\mathbb{R}^N$, and

$$\int_{(0,1)\times C} \phi^\varepsilon(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C|\min(c,1).$$

From standard elliptic estimates, and from the monotonicity of ϕ^ε with respect to s, one states that $\phi^\varepsilon(s,x) \to \phi^\varepsilon_\pm(x)$ as $s \to \pm\infty$ in $C^2(\overline{C})$. Moreover, $\phi^\varepsilon_\pm$ are L-periodic and satisfy

$$\nabla\cdot(A(x)\nabla\phi^\varepsilon_\pm) + f(x,\phi^\varepsilon_\pm) = 0 \text{ in } \overline{C},$$

with $0 \leq \phi^\varepsilon_\pm(x) \leq p(x)$ in $\overline{C}$.

But, as one has said before, from Theorems 2.1 and 2.3 of [17], the former equation, together with the bounds $0 \leq \phi^{\varepsilon}_{\pm}(x) \leq p(x)$, admits exactly two nonnegative solutions, which are 0 and p. Since ϕ^{ε} is nondecreasing in s, and from (51), one has

$$\int_C \phi^{\varepsilon}_{+}(x)dx \geq \frac{(p^-)^2}{2p^+}|C| \min(c,1) > 0, \tag{53}$$

and

$$\int_C \phi^{\varepsilon}_{-}(x)dx \leq \frac{(p^-)^2}{2p^+}|C| \min(c,1) < \int_C p(x)dx. \tag{54}$$

From (53) one deduces that $\phi^{\varepsilon}_{+} \equiv p$ and from (54) one has $\phi^{\varepsilon}_{-} \equiv 0$.

Coming back to the original variables (t,x), one defines $u^{\varepsilon}(t,x) = \phi^{\varepsilon}(x \cdot e + ct, x)$. As it was done in the proof of (36) and Lemma 6, it follows from (52) and from the limiting behavior of ϕ^{ε} as $s \to \pm\infty$ that u^{ε} satisfies the estimates (39), independently of ε. As it was done in subsection 2.4, there exists a function $u \in H^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$ such that (up to the extraction of a subsequence), $u^{\varepsilon} \rightharpoonup u$ weakly in $H^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$. From parabolic regularity, u is then a classical solution of

$$\begin{cases} u_t - \nabla_x \cdot (A(x)\nabla_x u) - f(x,u) &= 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ 0 \leq u \leq p, \; u_t &\geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N. \end{cases}$$

Moreover, as it was done in subsection 2.4, one still has $u(t + \frac{k \cdot e}{c}, x) = u(t, x+k)$ in $\mathbb{R} \times \mathbb{R}^N$ for all $k \in \prod_{i=1}^N L_i\mathbb{Z}$. Furthermore, u satisfies

$$\int_{\{0 < x \cdot e + ct < 1, \; x \in C\}} u(t,x)dtdx = \frac{(p^-)^2}{2cp^+}|C| \min(c,1).$$

One deduces from standard parabolic estimates and from the monotonicity of u in t, that $u(t,x) \to u^{\pm}(x)$ locally in x as $t \to \pm\infty$, and that $u^{\pm}$ solve $\nabla \cdot (A(x)\nabla u^{\pm}) + f(x,u^{\pm}) = 0$ in $\mathbb{R}^N$. Moreover, $0 \leq u^{\pm} \leq p$. From the monotonicity of u with respect to t, one can also assert that

$$\int_C u^{+}(x)dx \geq \frac{(p^-)^2}{2cp^+}|C| \min(c,1) > 0,$$

and

$$\int_C u^{-}(x)dx \leq \frac{(p^-)^2}{2cp^+}|C| \min(c,1) < \int_C p(x)dx.$$

Therefore using the same argument as for $\phi^{\varepsilon}_{\pm}$, one concludes that $u^+ \equiv p$ and $u^- \equiv 0$.

Finally, one deduces from the (t,x)-periodicity of u and the positivity of c that $u(t,x) \to 0$ as $x \cdot e \to -\infty$ and $u(t,x) - p(x) \to 0$ as $x \cdot e \to +\infty$, locally in t. Thus (c,u) is a classical solution of (10-11). Moreover, since $u_t \geq 0$, the strong parabolic maximum principle yields that u is increasing in t.

That completes the proof of Proposition 5. □

3 Monotonicity of the solutions

We are going to establish the monotonicity result in Theorem 1, namely, each solution (c,u) of (10-11) is such that u is increasing with respect to t. This will enable us to define a minimal speed c^* in the next section.

One first establishes the following lemma, which is close to Lemma 6.5 of [13]. Nevertheless, its proof does not use the fact that f has a given sign (which is not true in general) and clearly uses the property that 0 is an unstable solution of the stationary problem.

Lemma 7 *Let (c,u) be a classical solution of (10-11). Then $c>0$ and*

$$0<\Lambda\ :=\liminf_{t\to-\infty,\ x\in\overline{C}}\frac{u_t(t,x)}{u(t,x)}<+\infty.$$

PROOF. Let us first prove that $c>0$. Set $\phi(s,x)=u\left(\dfrac{s-x\cdot e}{c},x\right)$. From standard parabolic estimates, $u_t(t,x)\to 0$ as $t\to\pm\infty$, $\nabla_x u(t,x)\to 0$ as $t\to-\infty$ and $\nabla_x u(t,x)\to\nabla_x p(x)$ as $t\to+\infty$. Therefore $\phi_s(s,x)\to 0$ as $s\to\pm\infty$, $\nabla_x\phi(s,x)\to 0$ as $s\to-\infty$ and $\nabla_x\phi(s,x)\to\nabla_x p(x)$ as $s\to+\infty$. Then, arguing as in Lemma 5 (case 1), one can prove that

$$c\int_{\mathbb{R}\times C}(\phi_s)^2=-E(p)>0.$$

Therefore $c>0$.

Next, as it was done in [13] (Lemma 6.5), one can assert, using standard interior estimates, Harnack type inequalities and the (t,x)-periodicity of u, that u_t/u and $\nabla u/u$ are globally bounded. Let Λ be defined as in the stating of the above lemma. Then Λ is a finite real number.

Let (t_n,x_n) be a sequence in $\mathbb{R}\times\overline{C}$ such that $t_n\to-\infty$ and

$$u_t(t_n,x_n)/u(t_n,x_n)\to\Lambda \text{ as } n\to+\infty.$$

Up to the extraction of some subsequence, one can assume that $x_n\to x_\infty\in\overline{C}$ as $n\to+\infty$. Now set

$$w_n(t,x)=\frac{u(t+t_n,x)}{u(t_n,x_n)}.$$

From the boundedness of u_t/u and $\nabla u/u$, one can assert that the functions w_n are locally bounded. Moreover, they satisfy

$$\partial_t w_n-\nabla\cdot(A(x)\nabla w_n)-\frac{f(x,u(t+t_n,x))}{u(t+t_n,x)}w_n=0 \text{ in } \mathbb{R}\times\mathbb{R}^N.$$

From standard parabolic estimates, the positive functions w_n converge, up to the extraction of some subsequence, to a function w_∞, which is a nonnegative classical solution of

$$\partial_t w_\infty-\nabla\cdot(A(x)\nabla w_\infty)-f_u(x,0)w_\infty=0 \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

since $u(t,x)\to 0$ as $t\to-\infty$ locally in x. Moreover, $w_\infty(0,x_\infty)=1$, thus w_∞ is positive from the strong parabolic maximum principle. One also has $w_\infty(t+k\cdot e/c,x)=w_\infty(t,x+k)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and for all $k\in\prod_{i=1}^N L_i\mathbb{Z}$.

Then using the arguments of the Lemma 6.5 of [13], one can check that $w_\infty(t,x)e^{-\Lambda t}$ does not depend on t. Indeed, one clearly has $(w_\infty)_t/w_\infty\geq\Lambda$, and $(w_\infty)_t(0,x_\infty)=\Lambda w_\infty(0,x_\infty)$ from the definition of (t_n,x_n). Therefore, the function $z=(w_\infty)_t/w_\infty$ satisfies

$$\partial_t z-\nabla\cdot(A(x)\nabla z)-2\frac{\nabla w_\infty}{w_\infty}\cdot\nabla z=0 \ \text{ in } \mathbb{R}\times\mathbb{R}^N$$

with $z\geq\Lambda$ and $z(0,x_\infty)=\Lambda$, and $\nabla w_\infty/w_\infty$ is bounded. The strong maximum principle yields $z\equiv\Lambda$; in other words, $w_\infty(t,x)e^{-\Lambda t}$ does not depend on t.

Therefore the $C^2(\mathbb{R}^N)$ function $\psi(x) = w_\infty(0,x)e^{-\Lambda(x\cdot e)/c}$ is positive and L-periodic. Moreover, it satisfies

$$-L_{c,\lambda}\psi = 0, \tag{55}$$

where one has set $\lambda = \Lambda/c$, and

$$\begin{array}{rcl} -L_{c,\lambda}\psi & = & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda(eA(x)\nabla\psi) \\ & & -[\lambda^2 eA(x)e + \lambda\nabla\cdot(A(x)e) - \lambda c + f_u(x,0)]\psi. \end{array}$$

Now, from [13] (Proposition 5.7.1), one knows that for all λ and c in $\mathbb{R}$, there exists a unique $\mu_c(\lambda) \in \mathbb{R}$ and a unique positive function $\psi_\lambda \in C^2(\mathbb{R}^N)$ such that

$$\left\{ \begin{array}{l} -L_{c,\lambda}\psi_\lambda = \mu_c(\lambda)\psi_\lambda \text{ in } \mathbb{R}^N, \\ \psi_\lambda \text{ is L-periodic, } \ \|\psi_\lambda\|_\infty = 1. \end{array} \right. \tag{56}$$

That allows us to define the function $\lambda \mapsto \mu_c(\lambda)$. Let us show that it is concave. First, using the result 2) of Proposition 5.7 in [13], one has

$$\mu_c(\lambda) = \max_{\phi\ \in\ E}\ \inf_{\mathbb{R}^N}\ \frac{-L_{c,\lambda}\phi}{\phi},$$

where $E = \left\{\phi\ \in\ C^2(\mathbb{R}^N),\ \phi > 0, \phi\ \text{ is } L-\text{periodic}\right\}$. Let E'_λ be the set defined by

$$E'_\lambda = \left\{\phi\ \in\ C^2(\mathbb{R}^N),\ \exists\ \Upsilon\ \in\ E \text{ with } \phi(x) = e^{\lambda x\cdot e}\Upsilon\right\}.$$

Then, $\mu_c(\lambda) = c\lambda + h(\lambda)$ with

$$h(\lambda) = \max_{\phi\ \in\ E'_\lambda}\ \inf_{\mathbb{R}^N}\ \left\{\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0)\right\}.$$

Our aim is to show that h is concave. Let λ_1, $\lambda_2 \in \mathbb{R}$ and $t \in [0,1]$. Set $\lambda = t\lambda_1 + (1-t)\lambda_2$. One only has to show that $h(\lambda) \geq th(\lambda_1) + (1-t)h(\lambda_2)$. Let ϕ_1 and ϕ_2 be two arbitrary chosen functions in E'_{λ_1} and E'_{λ_2} respectively, and set $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = tz_1 + (1-t)z_2$ and $\phi = e^z$. It easily follows that $\phi \in E'_\lambda$. Therefore

$$h(\lambda) \geq \inf_{\mathbb{R}^N}\ \left\{\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0)\right\}.$$

Moreover,

$$\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0) = -\nabla\cdot(A(x)\nabla z) - \nabla zA(x)\nabla z - f_u(x,0),$$

and

$$\begin{array}{rcl} \nabla zA(x)\nabla z & = & t\nabla z_1A(x)\nabla z_1 + (1-t)\nabla z_2A(x)\nabla z_2 \\ & & -t(1-t)(\nabla z_1 - \nabla z_2)A(x)(\nabla z_1 - \nabla z_2) \\ & \leq & t\nabla z_1A(x)\nabla z_1 + (1-t)\nabla z_2A(x)\nabla z_2, \end{array}$$

since $0 \leq t \leq 1$.

As a consequence,

$$\begin{array}{rcl}\dfrac{-\nabla\cdot(A(x)\nabla\phi)}{\phi}-f_u(x,0) & \geq & t[-\nabla\cdot(A(x)\nabla z_1)-\nabla z_1 A(x)\nabla z_1-f_u(x,0)]\\ & & +(1-t)[-\nabla\cdot(A(x)\nabla z_2)-\nabla z_2 A(x)\nabla z_2-f_u(x,0)]\\ & \geq & t\left(\dfrac{-\nabla\cdot(A(x)\nabla\phi_1)}{\phi_1}-f_u(x,0)\right)\\ & & +(1-t)\left(\dfrac{-\nabla\cdot(A(x)\nabla\phi_2)}{\phi_2}-f_u(x,0)\right).\end{array}$$

Thus,

$$h(\lambda)\geq t\inf_{\mathbb{R}^N}\left(\frac{-\nabla\cdot(A(x)\nabla\phi_1)}{\phi_1}-f_u(x,0)\right)+(1-t)\inf_{\mathbb{R}^N}\left(\frac{-\nabla\cdot(A(x)\nabla\phi_2)}{\phi_2}-f_u(x,0)\right),$$

and, since ϕ_1 and ϕ_2 were arbitrarily chosen, one bets that $h(\lambda)\geq th(\lambda_1)+(1-t)h(\lambda_2)$. Therefore h is concave. This implies that h is continuous. Thus $\lambda\mapsto\mu_c(\lambda)=c\lambda+h(\lambda)$ is continuous and concave.

Next, let us show that $\mu_c(0)<0$. By definition, $\mu_c(0)$ is the first eigenvalue of the linear problem $-\nabla\cdot(A(x)\nabla\psi)\psi-f_u(x,0)\psi$ with L-periodicity conditions. From the hypothesis for conservation, it follows that $\mu_c(0)<0$.

Finally, it remains to show that $\mu_c'(0)>0$. For each $\lambda\in\mathbb{R}$, consider the positive and L-periodic eigenfunction $\psi_\lambda\in C^2(\mathbb{R}^N)$ for problem (56), associated to the eigenvalue $\mu_c(\lambda)$. By definition, it satisfies the equation

$$\begin{array}{rcl}-\nabla\cdot(A(x)\nabla\psi_\lambda)-2\lambda(eA(x)\nabla\psi_\lambda) & & \\ -(\lambda^2 eA(x)e+\lambda\nabla\cdot(A(x)e)-\lambda c+f_u(x,0))\psi_\lambda & = & \mu_1(\lambda)\psi_\lambda.\end{array}$$

Multiply this equation by ψ_0, and integrate it by parts over C. One obtains, using the L-periodicity of ψ_λ and ψ_0, and since the matrix field $A(x)$ is symmetric,

$$\begin{array}{rcl}-\displaystyle\int_C\psi_\lambda\nabla\cdot(A(x)\nabla\psi_0)-\lambda\int_C[\nabla\cdot(A(x)e\psi_\lambda)+eA(x)\nabla\psi_\lambda]\psi_0 & & \\ -\displaystyle\int_C\left[(\lambda^2 eA(x)e-\lambda c)\psi_\lambda+f_u(x,0)\psi_\lambda\right]\psi_0 & = & \mu_c(\lambda)\displaystyle\int_C\psi_\lambda\psi_0.\end{array}\qquad(57)$$

Multiplying by ψ_λ the equation satisfied by ψ_0, one obtains,

$$-\int_C[\psi_\lambda\nabla\cdot(A(x)\nabla\psi_0)+f_u(x,0)\psi_\lambda\psi_0]=\mu_c(0)\int_C\psi_\lambda\psi_0.\qquad(58)$$

Substituting (58) into (57), and dividing by λ, one gets

$$\begin{array}{rcl}-\displaystyle\int_C[\nabla\cdot(A(x)e\psi_\lambda)+eA(x)\nabla\psi_\lambda]\psi_0 & & \\ -\displaystyle\int_C(\lambda eA(x)e-c)\psi_\lambda\psi_0 & = & \dfrac{\mu_c(\lambda)-\mu_c(0)}{\lambda}\displaystyle\int_C\psi_\lambda\psi_0.\end{array}\qquad(59)$$

Now, take an arbitrary sequence $\lambda_n\to 0$. Since $\mu_c(\lambda_n)\to\mu_c(0)$, standard elliptic estimates, and Sobolev injections imply, up to the extraction of some subsequence, that the functions ψ_{λ_n} converge locally (and therefore uniformly by L-periodicity) in $C^{2,\beta}$ (for all $0\leq\beta<1$) to a nonnegative function ψ^0 such that $\|\psi^0\|_\infty=1$, ψ^0 is L-periodic and satisfies

$$-\nabla\cdot(A(x)\nabla\psi^0)-f_u(x,0)\psi^0=\mu_c(0)\psi^0.$$

From strong elliptic maximum principle, it follows that $\psi^0 > 0$, and by uniqueness (up to normalization), $\psi^0 = \psi_0$, and the whole family ψ_{λ_n} converges to ψ_0 as $n \to +\infty$. Therefore, passing to the limit $\lambda \to 0$ in (59), one obtains that μ_c is differentiable at 0, and

$$-\int_C [\nabla \cdot (A(x)e\psi_0) + eA(x)\nabla\psi_0]\psi_0 + c\int_C \psi_0^2 = \mu_c'(0)\int_C \psi_0^2.$$

From the L-periodicity of ψ_0, and since the matrix field $A(x)$ is symmetric, one has

$$\int_C [\nabla \cdot (A(x)e\psi_0) + eA(x)\nabla\psi_0]\psi_0 = \int_C [eA(x)\nabla\psi_0 - A(x)e \cdot \nabla\psi_0]\psi_0 = 0,$$

whence

$$\mu_c'(0) = c > 0.$$

Therefore, one has shown that $\lambda \mapsto \mu_c(\lambda)$ is concave, with $\mu_c(0) < 0$ and $\mu_c'(0) > 0$. Moreover, coming back to our solution ψ of (55), one has $\mu_c(\Lambda/c) = 0$. Therefore $\Lambda > 0$, and the lemma is proved. □

One can now turn to the proof of the monotonicity result in Theorem 1. Set $\phi(s,x) = u\left(\frac{s - x \cdot e}{c}, x\right)$. Then

$$u_t(t,x)/u(t,x) = c\phi_s(x \cdot e + ct, x)/\phi(x \cdot e + ct, x).$$

One knows from Lemma 7 that $c > 0$ and

$$\liminf_{t \to -\infty,\ x \in \overline{C}} \frac{u_t(t,x)}{u(t,x)} > 0.$$

Therefore,

$$\liminf_{s \to -\infty,\ x \in \mathbb{R}^N} \frac{\phi_s(s,x)}{\phi(s,x)} > 0$$

and, from the L-periodicity of ϕ with respect to x, one can deduce that there exists $\overline{s} \in \mathbb{R}$ such that

$$\forall\ s \le \overline{s},\ \forall\ x \in \mathbb{R}^N,\ \phi_s(s,x) > 0.$$

Moreover $\inf_{s \ge \overline{s},\ x \in \mathbb{R}^N} \phi(s,x) > 0$ and $\phi(-\infty, x) = 0$ uniformly in $x \in \mathbb{R}^N$. As a consequence, there exists $B \in \mathbb{R}$ such that $-B \le \overline{s}$ and

$$\forall\ \tau \ge 0,\ \forall\ s \le -B,\ \forall\ x \in \mathbb{R}^N,\ \phi(s,x) \le \phi^\tau(s,x) \tag{60}$$

where one has defined $\phi^\tau(s,x) = \phi(s+\tau, x)$. One can assume that $B \ge 0$.

Fix now any $\tau \ge 0$. Set

$$\lambda^* = \inf\left\{\lambda,\ \lambda\phi^\tau \ge \phi \text{ in } [-B, +\infty) \times \mathbb{R}^N\right\}.$$

The real λ^* is well defined since ϕ is bounded and $\inf_{s \ge -B,\ x \in \mathbb{R}^N} \phi^\tau(s,x) > 0$.

Assume $\lambda^* > 1$. Since $\phi(s,x) \to p(x) > 0$ as $s \to +\infty$ uniformly in x, with p bounded from below, and since ϕ is L-periodic in x, there exists a point $(s_0, x_0) \in [-B, +\infty) \times \overline{C}$ such that $\lambda^*\phi^\tau(s_0, x_0) = \phi(s_0, x_0)$.

Furthermore, $\lambda^*\phi^\tau \geq \phi$ in $\in [-B, +\infty) \times \mathbb{R}^N$ by continuity and in $(-\infty, -B] \times \mathbb{R}^N$ by (60) and because $\lambda^* > 1$. Coming back to the original variables (t, x), set $z(t,x) = \lambda^*\phi^\tau(x\cdot e + ct, x) - \phi(x \cdot e + ct, x)$. Then $z \geq 0$ in $\mathbb{R} \times \mathbb{R}^N$, moreover, z satisfies the following equation :

$$z_t - \nabla \cdot (A(x)\nabla z) = \lambda^* f(x, \phi^\tau) - f(x, \phi).$$

Therefore, using (7), one obtains

$$f(x, \lambda^*\phi^\tau) \leq \lambda^* f(x, \phi^\tau).$$

Thus one has

$$z_t - \nabla \cdot (A\nabla z) \geq f(x, \lambda^*\phi^\tau) - f(x, \phi).$$

Therefore, there exists a bounded function b such that

$$z_t - \nabla \cdot (A(x)\nabla z) + b(x)z \geq 0. \tag{61}$$

Furthermore, since $\lambda^*\phi_\tau(s_0, x_0) = \phi(s_0, x_0)$, setting $t_0 = \frac{s_0 - x_0\cdot e}{c}$, one has $z(t_0, x_0) = 0$. Besides, z is nonnegative, and satisfies (61) ; therefore, from the strong parabolic maximum principle, one has $z(t, x) = 0$ for all $t \leq t_0$ and $x \in \mathbb{R}^N$, whence $z(t, x) \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$ since $z\left(t + \frac{k\cdot e}{c}, x\right) = z(t, x + k)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $k \in \prod_{i=1}^N L_i\mathbb{Z}$. One gets a contradiction since $z(t, x) \to (\lambda^* - 1)p(x) > 0$ as $t \to +\infty$.

Thus $\lambda^* \leq 1$ for all $\tau \geq 0$, whence $\phi^\tau \geq \phi$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\tau \geq 0$. One therefore gets that ϕ is nondecreasing with respect to s, and u is nondecreasing in t because $c > 0$. Finally, with the same arguments as above, one can prove, using the strong parabolic maximum principle, that u is increasing in t. That concludes the proof of the monotonicity result in Theorem 1. □

Remark 1 Notice that this monotonicity result especially implies that any solution (c, u) of (10-11) is such that $0 < u(t, x) < p(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

4 The minimal speed c^*

This section is devoted to the proof of the existence of a minimal speed of propagation of the pulsating fronts, and some properties of this minimal speed with respect to the nonlinearity f.

4.1 Existence of a positive minimal speed c^*

As it was proved in Lemma 7, any solution (c, u) of (10-11) is such that $c > 0$. In order to complete the proof of the first part of Theorem 1 and to obtain the existence of a $c^* > 0$ such that there exists a solution of (10-11) if and only if $c \geq c^*$.

Proposition 6 *There exists $c^* > 0$ such that, for $c \geq c^*$, there exists a solution u of (10-11), while no solution exists for $c < c^*$.*

PROOF. First, assume by contradiction that there exists a sequence $c_n \to 0^+$ and some classical functions u_n such that (c_n, u_n) is a solution of (10-11).

Again, Let $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$, be such that $x_0 \cdot e > 0$. One can assume that $u_n(0, x_0) = \dfrac{p^-}{2}$.

From standard parabolic estimates, the positive functions u_n converge locally uniformly, up to the extraction of some subsequence, to a nondecreasing (in t) function u, which is a classical solution of

$$\partial_t u - \nabla\cdot(A(x)\nabla u) = f(x,u) \ \in \mathbb{R}\times\mathbb{R}^N.$$

Moreover, u satisfies $0 \le u \le p$ and one has $u(0,x_0) = \dfrac{p^-}{2}$.

Since u is nondecreasing in t, one can define $u^+(x) =: \lim_{t\to+\infty} u(t,x)$, and from standard elliptic estimates, u^+ satisfies $\nabla\cdot(A(x)\nabla u^+) + f(x,u^+) = 0$. Moreover $0 \le u^+ \le p$. Hence, as already said (using Theorems 2.1 and 2.3 of [17]), $u^+ \equiv 0$ or $u^+ \equiv p$. But for every $B > 0$, for n large enough, $u_n(B,0) \le u_n(\frac{x_0\cdot e}{c_n},0) = u_n(0,x_0) = \frac{p^-}{2}$. Therefore $u^+(0) \le \frac{p^-}{2}$. Thus $u^+ \equiv 0$. But since u is nondecreasing and $u(0,x_0) = \frac{p^-}{2}$, $u^+(0) \ge \frac{p^-}{2}$, which is contradictory with the preceding result.

On the other hand, the arguments used in Proposition 5 actually imply that, if (c_0,u_0) is a solution of (10-11) with $c_0 > 0$ and $(u_0)_t > 0$, then there is a solution (c,u) of (10-11) for each $c > c_0$.

Using Lemma 7, one concludes that there exits $c^* > 0$ such that for all $c > c^*$, there exists a solution u of (10-11), while no solution exists for $c < c^*$.

In particular, there exists a sequence (c_n,u_n) of solutions of (10-11), such that $c_n \to c^*$ as $n\to+\infty$, with $c_n > c^*$. As it was done in the first part of the proof of this Proposition assume that $u_n(0,x_0) = \frac{p^-}{2}$. From standard parabolic estimates, u_n converge locally uniformly in (up to the extraction of some subsequence), to a classical solution u^* of

$$\partial_t u^* - \nabla\cdot(A(x)\nabla u^*) = f(x,u^*) \ \in \mathbb{R}\times\mathbb{R}^N,$$

with $0 \le u^* \le p$, and $u^*_t \ge 0$. Moreover, $u^*(0,x_0) = \frac{p^-}{2}$. Using the same arguments as those of the beginning of this proof, one concludes that $\lim_{t\to-\infty} u^*(t,x) = 0$ and $\lim_{t\to+\infty} u^*(t,x) = p(x)$, locally in x. Furthermore, by passing to the limit, $u^*\left(t+\frac{k\cdot e}{c^*},x\right) = u^*(t,x+k)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and $k\in\prod_{i=1}^N L_i\mathbb{Z}$. Finally, the strong maximum principle, with $u^*_t \ge 0$, gives us that u^* is increasing in t. □

4.2 Characterization of c^*

This section is devoted to the proof of the variational characterization of the minimal speed c^*. Notice first that the assumption (7) implies that

$$\forall\ x\in\mathbb{R}^N,\ \forall\ u\ge 0,\ f(x,u) \le f_u(x,0)u. \tag{62}$$

Let us define

$$c_0^* = \inf\left\{c\in\mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c(\lambda) = 0\right\},$$

where $\mu_c(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{aligned} -L_{c,\lambda}\psi = \quad & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda\ eA(x)\nabla\psi \\ & -[\lambda^2 eA(x)e + \lambda\nabla\cdot(A(x)e) - \lambda c + f_u(x,0)]\psi, \end{aligned}$$

with L-periodicity conditions.

Proposition 7 *One has $c^* = c_0^*$.*

The proof is divided into several lemmas.

Lemma 8 *The real number c_0^* does exist and $0 \leq c_0^* \leq c^*$.*

PROOF. Let $c \geq c^*$, and (c,u) be a solution of (10-11). Then, arguing as in the proof of Lemma 7, one obtains a positive function ψ, satisfying (55) with $\lambda = \Lambda/c > 0$. In other words, $\mu_c(\lambda) = 0$. That yields $c_0^* \leq c^*$.

Moreover, using the concavity of $\lambda \mapsto \mu_c(\lambda)$, which has been shown in the proof of Lemma 7, together with $\mu_c(0) < 0$ and $(\mu_c)'(0) = c$, one immediately gets that if $c < 0$, then $\mu_c(\lambda) < 0$ for all $\lambda > 0$. Therefore 0 is a lower bound of the set $\{c \in \mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c(\lambda) = 0\}$. □

From Lemma 8 and Proposition 6, the next lemma follows :

Lemma 9 *For all $c < c_0^*$, problem (10-11) has no solution (c,u).*

Now, for all $\varepsilon > 0$, let us define

$$c_\varepsilon^* = \inf\left\{c \in \mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c^\varepsilon(\lambda) = 0\right\},$$

where $\mu_c^\varepsilon(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{aligned} -L_{c,\lambda}^\varepsilon \psi = \quad & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda e A(x)\nabla\psi \\ & -(eA(x)e + \varepsilon)\lambda^2\psi - \lambda\nabla\cdot(A(x)e)\psi + \lambda c\psi - f_u(x,0)\psi, \end{aligned}$$

with L-periodicity conditions.

First, using a result of [13] (Proposition 5.7.2), one obtains that

$$\mu_c^\varepsilon(\lambda) = \max_{\phi\in E}\ \inf_{\mathbb{R}^N}\ \frac{-L_{c,\lambda}^\varepsilon \phi}{\phi},$$

where $E = \left\{\phi \in C^2(\mathbb{R}^N),\ \phi > 0,\ \phi \text{ is L-periodic}\right\}$.

Set

$$j(\lambda) = \max_{\phi\in E}\ \inf_{x\in\mathbb{R}^N}\ \left\{\frac{-\nabla\cdot(A(x)\nabla\phi) - 2\lambda eA(x)\nabla\phi}{\phi} - \lambda^2 eA(x)e - \lambda\nabla\cdot(A(x)e) - f_u(x,0)\right\}.$$

Then,

$$\mu_c^\varepsilon(\lambda) = j(\lambda) + \lambda c - \varepsilon\lambda^2 = \mu_c(\lambda) - \varepsilon\lambda^2.$$

Lemma 10 *The real number c_ε^* does exist for all $\varepsilon > 0$, and $c_\varepsilon^* \geq 0$.*

PROOF. Let ε be fixed. Let λ be given. Since $\mu_c^\varepsilon(\lambda) = j(\lambda) + \lambda c - \varepsilon\lambda^2$, there exists $c > 0$ large enough such that $\mu_c^\varepsilon(\lambda) > 0$. Since $\mu_c^\varepsilon(0) = \lambda_1 < 0$ (λ_1 is the first eigenvalue of $-\nabla\cdot(A(x)\nabla\psi) - f_u(x,0)\psi$ with L-periodicity conditions), and since $\lambda \mapsto \mu_c^\varepsilon(\lambda) = \mu_c(\lambda) - \varepsilon\lambda^2$ is concave, whence continuous, one gets the existence of λ' such that $\mu_c^\varepsilon(\lambda') = 0$. Therefore $c_\varepsilon^* < +\infty$ for all $\varepsilon > 0$.

Moreover, as it was done in Lemma 8, one easily sees that 0 is a lower bound of the set $\{c \in \mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c^\varepsilon(\lambda) = 0\}$. □

Let us now show that

Lemma 11 *For all $c > c_\varepsilon^*$, there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$.*

PROOF. Let c be s.t. $c > c_\varepsilon^*$. From the definition of c_ε^*, one knows that there exists a sequence (c_n) such that $c_n \to c_\varepsilon^*$ as $n \to +\infty$ and, for each n, there is $\lambda_n > 0$ with $\mu_c^\varepsilon(\lambda_n) = 0$. Therefore, there exists N such that $c_N < c$. One has $\mu_c^\varepsilon(\lambda_N) = \mu_{c_N}^\varepsilon(\lambda_N) + (c - c_N)\lambda_N > 0$. Using the same argument than this of Lemma 10, one deduces that there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. □

Next, let us prove that

Lemma 12 *One has $c_\varepsilon^* \to c_0^*$ as $\varepsilon \to 0$.*

PROOF. First, one observes that $c_\varepsilon^* \geq c_0^*$ for all $\varepsilon > 0$. Indeed, for $c > c_\varepsilon^*$, there exists, from Lemma 11, $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. Thus, since $\mu_c(\lambda) > \mu_c^\varepsilon(\lambda) = 0$, arguing as in Lemma 11, one easily sees that there exists $\lambda_0 > 0$ such that $\mu_c(\lambda_0) = 0$.

Next, let us show that for any $c > c_0^*$, there exists ε_0 such that $c \geq c_\varepsilon^*$ for all $\varepsilon < \varepsilon_0$.

Indeed, one deduces from Lemma 11, adapted to c_0^*, that for each $c > c_0^*$, there exists $\lambda_1 > 0$ such that $\mu_{\frac{c+c_0^*}{2}}(\lambda_1) = 0$. Then

$$\mu_c^\varepsilon(\lambda_1) = \mu_{\frac{c+c_0^*}{2}}(\lambda_1) + \frac{c - c_0^*}{2}\lambda_1 - \lambda_1^2\varepsilon.$$

Thus, $\mu_c^\varepsilon(\lambda_1) = \frac{c-c_0^*}{2}\lambda_1 - \lambda_1^2\varepsilon$. Hence, for ε small enough, $\mu_c^\varepsilon(\lambda_1) > 0$. Therefore, there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. Finally, from the definition of c_ε^*, one deduces that $c \geq c_\varepsilon^*$, and the lemma is proved. □

Let us now turn to the

PROOF of Proposition 7. Let c be such that $c > c_0^*$. Then, from Lemma 12, one knows that for ε small enough, $c > c_\varepsilon^*$. Therefore, from Lemma 11, there exist $\lambda > 0$ and $\psi > 0$ L-periodic, depending on ε, and such that

$$-L_{c,\lambda}^\varepsilon \psi = 0. \tag{63}$$

Now, set $\phi^1(s,x) := \psi(x)e^{\lambda s}$, for all $(s,x) \in \mathbb{R} \times \mathbb{R}^N$, and let L_ε be defined as in the proof of Proposition 5. Then,

$$\begin{aligned} L_\varepsilon \phi^1 = \quad & \{\, \nabla \cdot (A(x)\nabla\psi) + 2\lambda e A(x)\nabla\psi \\ & +(eA(x)e + \varepsilon)\lambda^2\psi + \lambda\nabla\cdot(A(x)e)\ \psi - \lambda c\psi\} \, e^{\lambda s}, \end{aligned}$$

and, since ψ satisfies (63), one has

$$L_\varepsilon \phi^1 = -f_u(x,0)\phi^1.$$

Therefore, using (62), one obtains,

$$L_\varepsilon\phi^1 + f(x,\phi^1) = f(x,\phi^1) - f_u(x,0)\phi^1 \leq 0. \tag{64}$$

Moreover, ϕ^1 is increasing in s and L-periodic with respect to x.

Now, with the notations of Section 2.1, and as it was proved in Lemma 1, there exists a solution $\phi_\tau \in C^2\left(\overline{\Sigma_a}\right)$ of the following problem :

$$\begin{cases} L_\varepsilon\phi_\tau + f(x,\phi_\tau) = 0 \text{ in } \Sigma_a, \\ \phi_\tau \text{ is L-periodic in } x, \\ \phi_\tau(-a,x) = \min\left\{ \inf\limits_{y\in\overline{C}} \phi^1(-a+\tau,y), p^- \right\} p(x)/p^+, \\ \phi_\tau(a,x) = \min\{\phi^1(a+\tau,x), p(x)\} \end{cases} \tag{65}$$

First, following the proof of Proposition 5, one obtains that

$$\forall\ (s,x)\ \in\ \Sigma_a,\ \phi_\tau(-a,x) < \phi_\tau(s,x). \tag{66}$$

Next, in the case $\phi^1(a+\tau,x) > p(x)$ for all $x \in \overline{C}$, one obtains in the same way that

$$\forall\ (s,x)\ \in\ \Sigma_a,\ \phi_\tau(a,x) = p(x) > \phi_\tau(s,x). \tag{67}$$

Now, in the other case, one has $\phi_\tau(a,x) = \min\left\{\phi^1(a+\tau,x), p(x)\right\}$. But, since $\phi^1(s,x) \to +\infty$ as $s \to +\infty$, one has, for k large enough, $\phi^1(s+\tau+k,x) > \phi_\tau(s,x)$ in $\overline{\Sigma_a}$. Let $\overline{k}$ be the smallest k such that the latter holds. It exists since $\phi^1(s,x) \to 0$ as $s \to -\infty$ and $\phi_\tau(s,x) > 0$ in $\overline{\Sigma_a}$. Assume $\overline{k} > 0$. By continuity, $\phi^1(s+\tau+\overline{k},x) \geq \phi_\tau(s,x)$ with equality at a point $(\overline{s},\overline{x}) \in \overline{\Sigma_a}$. Since ϕ^1 is increasing in s, $\phi^1(-a+\tau+\overline{k},x) > \phi^1(-a+\tau,x) \geq \phi_\tau(-a,x)$ in $\overline{C}$, and similarly, $\phi^1(a+\tau+\overline{k},x) > \phi^1(a+\tau,x) \geq \phi_\tau(a,x)$. Thus, $(\overline{s},\overline{x}) \in (-a,a)\times\overline{C}$ (one can assume this using the L-periodicity in x of ϕ^1 and ϕ_τ). But, from (64), it is found that $\phi^1(s+\tau+\overline{k},x)$ is a super-solution of (65). Therefore, the strong maximum principle implies that $\phi^1(s+\tau+\overline{k},x) \equiv \phi_\tau(s,x)$ in $\overline{\Sigma_a}$. One gets a contradiction with the boundary condition at $s = a$. As a consequence, $\overline{k} = 0$ and, one has

$$\forall\ (s,x) \in \overline{\Sigma_a},\ \phi_\tau(s,x) \leq \phi^1(s+\tau,x) \tag{68}$$

and, since ϕ^1 is increasing in s, it follows that $\phi_\tau(s,x) < \phi^1(a+\tau,x)$ in Σ_a.

As a conclusion, from (66), (67) and (68), one has

$$\forall\ (s,x) \in \Sigma_a,\ \phi_\tau(-a,x) < \phi_\tau(s,x) < \phi_\tau(a,x). \tag{69}$$

Using the same arguments as those of Proposition 5, it follows that ϕ_τ is increasing in s and is the unique solution of (65) in $C^2\left(\overline{\Sigma_a}\right)$. Moreover, since the boundary conditions for ϕ_τ at $s = \pm a$ are nondecreasing in τ, one can prove, as in Lemma 3, that the functions ϕ_τ are continuous with respect to τ in $C^2\left(\overline{\Sigma_a}\right)$ and nondecreasing in τ. But, since $\phi^1(-\infty,x) = 0$ and $\phi^1(+\infty,x) = +\infty$ in $\mathbb{R}^N$, it follows from (69) that $\phi_\tau \to 0$ as $\tau \to -\infty$ uniformly in $\overline{\Sigma_a}$ and that,

$$\forall\ \alpha > 0,\ \exists\ T,\ \forall \tau > T, \quad \phi_\tau > \frac{p^- p}{p^+} - \alpha \text{ in } \overline{\Sigma_a}.$$

Therefore, for each $a > 1$, there exists $\tau(a) \in \mathbb{R}$ such that $\phi^{\varepsilon,a} := \phi_{\tau(a)}$ solves (65) and satisfies

$$\int_{(0,1)\times C} \phi^{\varepsilon,a}(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C|\min(c,1).$$

Moreover $\phi_\tau(a,.)$ is bounded independently of a. Thus, letting $a_n \to +\infty$, from standard elliptic estimates, the functions ϕ^{ε,a_n} converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \leq \beta < 1$), up to the extraction of a subsequence, to a function ϕ^ε satisfying

$$\left\{\begin{array}{rcl} L_\varepsilon \phi^\varepsilon + f(x,\phi^\varepsilon) & = & 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi^\varepsilon \text{ is L-periodic w.r.t. } x. & & \end{array}\right.$$

Moreover, ϕ^ε is nonincreasing with respect to s, and satisfies

$$0 \leq \phi^\varepsilon(s,x) \leq p(x) \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

and

$$\int_{(0,1)\times C} \phi^\varepsilon(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C|\min(c,1).$$

Next, passing to the limit $\varepsilon \to 0$ and using the same arguments as those of the end of the proof of Proposition 5, one obtains a solution (c, u) of the problem (10-11).

But since c was chosen arbitrarily such that $c > c_0^*$, one concludes that there exists a solution (c, u) of (10-11) for all $c > c_0^*$. Next, using Lemma 9, one obtains that $c^* = c_0^*$. □

That completes the proof of Theorem 1.

4.3 Dependency of c^* with respect to the nonlinearity f

In this section, we study the dependency of c^*, with respect to the "shape" and the "size" of the nonlinearity f. This section is devoted to the proofs of Theorem 2 and Corolarry 1. In the whole section, one assumes that the matrix field $A(x) = A$ is constant, and one considers the problem (10-11), with a nonlinearity f such that $f_u(x, 0)$ is of the type $f_u(x, 0) = \mu(x) + B\nu(x)$, where B is a positive real number and ${}_m u$ and ν are periodic $C^{0,\alpha}$ functions.

It then follows from Theorem 2.8 of [17] that the function f satisfies the hypothesis for all $B > 0$ if $\displaystyle\int_C \mu \geq 0$ and $\displaystyle\int_C \nu \geq 0$ with $\nu \not\equiv 0$. It also follows from Theorem 2.8 of [17] that f satisfies the hypothesis for conservation for $B > 0$ large enough under the only assumption $\max \nu > 0$.

In the next propositions, we will make several uses of the following characterizations of c^* : first, from Theorem 1,

$$c^* = \inf \{c,\ \exists\ \lambda > 0 \text{ with } \mu_{c,B}(\lambda) = 0\}\,, \tag{70}$$

where $\mu_{c,B}(\lambda)$ is the principal eigenvalue of the elliptic operator

$$-L_{c,B,\lambda}\psi = -\nabla \cdot (A\nabla\psi) - 2\lambda Ae \cdot \nabla\psi - (\lambda^2\ eAe - \lambda c)\psi - (\mu(x) + B\nu(x))\psi,$$

on the set E of L-periodic C^2 functions. Furthermore, as it was said in [13] and in [93], for pulsating fronts in $\mathbb{R}^N$, the formula below is equivalent to the following one :

$$c^* = \min_{\lambda > 0} \frac{-k_\lambda(B)}{\lambda}, \tag{71}$$

where $k_\lambda(B)$ is the principal eigenvalue of the operator

$$-\mathcal{L}_{B,\lambda}\phi = -\nabla \cdot (A\nabla\phi) - 2\lambda Ae \cdot \nabla\phi - \lambda^2\ eAe\phi - (\mu(x) + B\nu(x))\phi,$$

acting on the same set E of functions ϕ. We call $\phi_{B,\lambda}$ be the principal eigenfunction associated to $k_\lambda(B)$. It satisfies

$$\begin{cases} -\mathcal{L}_{B,\lambda}\phi_{B,\lambda} = k_\lambda(B)\phi_{B,\lambda}, \\ \phi_{B,\lambda} \text{ is L-periodic, } \phi_{B,\lambda} > 0 \text{ in } \mathbb{R}^N, \\ \|\phi_{B,\lambda}\|_\infty = 1 \text{ (up to normalization).} \end{cases} \tag{72}$$

We are going to study the monotonicity of the function $B \mapsto c^* = c^*(B)$, as soon as the hypothesis for conservation if satisfied. One has the

Proposition 8 *Assume that $\mu = \mu_0 \geq 0$ is constant and assume that $\displaystyle\int_C \nu(x)dx \geq 0$ with $\max \nu > 0$. Then, the hypothesis for conservation is satisfied for all $B > 0$ and $c^*(B)$ is an increasing function of $B > 0$.*

PROOF. As already underlined at the beginning of this section, the hypothesis for conservation is satisfied for all $B > 0$.

As done in the proof of Lemma 7, one has

$$k_\lambda(B) = \max_{\phi \in E'_\lambda} \inf_{\mathbb{R}^N} \frac{-\nabla \cdot (A\nabla\phi)}{\phi} - \mu_0 - B\nu(x),$$

where E'_λ be the set defined by

$$E'_\lambda = \left\{\phi \in C^2(\mathbb{R}^N),\ \exists\ \Upsilon > 0,\ \Upsilon\ L\text{-periodic with } \phi(x) = e^{\lambda x \cdot e}\Upsilon\right\}.$$

Let $B_1,\ B_2 \in \mathbb{R}$ and $t \in [0,1]$. Set $B = tB_1 + (1-t)B_2$. Let ϕ_1 and ϕ_2 be two arbitrary chosen functions in E'_λ, and set $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = tz_1 + (1-t)z_2$ and $\phi = e^z$. It easily follows that $\phi \in E'_\lambda$. Therefore

$$k_\lambda(B) \geq \inf_{\mathbb{R}^N} \left\{\frac{-\nabla \cdot (A\nabla\phi)}{\phi} - \mu_0 - B\nu(x)\right\}.$$

Then, arguing as in the proof of Lemma 7, one obtains that

$$\begin{aligned} k_\lambda(B) \quad &\geq \quad t \inf_{\mathbb{R}^N} \left\{\frac{-\nabla \cdot (A\nabla\phi_1)}{\phi_1} - \mu_0 - B_1\nu(x)\right\} \\ &\quad +(1-t) \inf_{\mathbb{R}^N} \left\{\frac{-\nabla \cdot (A\nabla\phi_2)}{\phi_2} - \mu_0 - B_2\nu(x)\right\}, \end{aligned}$$

and, since ϕ_1 and ϕ_2 were arbitrary chosen, one has $k_\lambda(B) \geq tk_\lambda(B_1) + (1-t)k_\lambda(B_2)$. Therefore the function $B \mapsto k_\lambda(B)$ is concave. This also implies that this function is continuous.

Next, one easily sees that $k_\lambda(0) = -\lambda^2 eAe - \mu_0$, and that the associated eigenfunction $\phi_{0,\lambda}$ is equal to 1.

Now, let us calculate $k'_\lambda(0)$. Let $\phi_{B,\lambda}$ be the principal eigenfunction associated to $k_\lambda(B)$ defined in (72), and let us integrate by parts the equation (72) over C. Using the L-periodicity of $\phi_{B,\lambda}$, one obtains

$$-(\lambda^2 eAe + \mu_0) \int_C \phi_{B,\lambda} - B \int_C \nu(x)\phi_{B,\lambda} dx = k_\lambda(B) \int_C \phi_{B,\lambda}. \tag{73}$$

By continuity, one knows that $k_\lambda(B) \to k_\lambda(0)$ as $B \to 0$. Still arguing as in the proof of Lemma 7, one also knows that $\phi_{B,\lambda}$ converges in $C^{2,\beta}$ (for all $0 \leq \beta < 1$) to $\phi_{0,\lambda} \equiv 1$ as $B \to 0$. Then, dividing the equation (73) by B, one gets

$$\frac{k_\lambda(B) + \lambda^2 eAe + \mu_0}{B} \int_C \phi_{B,\lambda} = -\int_C \nu(x)\phi_{B,\lambda} dx.$$

Therefore, passing to the limit $B \to 0$, one obtains

$$k'_\lambda(0) = -\int_C \nu(x) dx.$$

In the case $\displaystyle\int_C \nu(x)dx > 0$, one has $k'_\lambda(0) < 0$. From the concavity of $B \mapsto k_\lambda(B)$, one deduces that this function is decreasing with respect to $B > 0$. Since this is true for all $\lambda > 0$, one concludes that the minimal speed $c^*(B)$ given in (71) is an increasing function of $B > 0$.

Similarly, if $\int_C \nu(x)dx = 0$, and $\max \nu > 0$, divide the equation (72) by $\phi_{B,\lambda}$ and integrate it by parts over C. By L-periodicity, one obtains

$$-\int_C \left[\frac{\nabla\phi_{B,\lambda} A \nabla\phi_{B,\lambda}}{\phi_{B,\lambda}^2}\right] - (\lambda^2 eAe + \mu_0)|C| - B\int_C \nu(x)dx = k_\lambda(B)|C|,$$

and, since $\phi_{B,\lambda}$ is not constant (because ν is not constant) and the matrix A is elliptic, one gets that $k_\lambda(B) < -(\lambda^2 eAe + \mu_0) = k_\lambda(0)$ for all $B > 0$. Hence, since $k'_\lambda(0) = 0$ and $k_\lambda(B)$ is concave in B, one concludes that $B \mapsto k_\lambda(B)$ is decreasing in $B > 0$. Finally, it follows that $c^*(B)$ is increasing in $B > 0$. □

The biological interpretation of this proposition is that increasing the amplitude of the favorableness of the environment increases the invasion's speed.

Remark 2 If one only assumes that $\max \nu > 0$, then the function f satisfies the hypothesis for conservation for $B > 0$ large enough. Furthermore, under the other assumptions of Proposition 8, the same arguments as above imply that the function $B \mapsto c^*(B)$ is an increasing function of B (for B large, as soon as the hypothesis forn conservation is satisfied).

In the next proposition, one assumes that f satisfies the hypothesis for conservation. As one has said above, it follows from Theorem 2.8 of [17] that it is true for all $B > 0$ if $\int_C \mu(x) \geq 0$ and $\int_C \nu(x) \geq 0$ with $\nu \not\equiv 0$; if one only has $\max \nu > 0$, it is true if B is large enough.

Proposition 9 *Assume that* $\max \nu > 0$ *and that the function* f *satisfies the hypothesis for conservation with* $f_u(x,0) = \mu(x) + B\nu(x)$*. Then*

$$c^*(B) \leq 2\sqrt{eAe \max(\mu + B\nu)}.$$

PROOF. Let us first observe that the definition of λ_1 in (9) and that the hypothesis for conservation ($\lambda_1 < 0$) imply that $\max(\mu + B\nu) > 0$.

Next, using the characterization of c^* given by (70), let us integrate by parts over C the equation $-L_{c,B,\lambda}\psi = 0$. Using the L-periodicity of ψ, one obtains the following inequalities :

$$\mu_{c,B}(\lambda) \geq -\lambda^2 eAe + c\lambda - \max(\mu + B\nu).$$

Therefore, if $c \geq 2\sqrt{eAe \max(\mu + B\nu)}$, there exists $\lambda_0 > 0$ such that $\mu_{c,B}(\lambda_0) \geq 0$. On the other hand, $\mu_{c,B}(0) = \lambda_1 < 0$ from the hypothesis for conservation. By continuity, it follows that there exists a solution $\lambda > 0$ of $\mu_{c,B}(\lambda) = 0$ as soon as $c \geq 2\sqrt{eAe \max(\mu + B\nu)}$. Thus, one finally has

$$c^*(B) \leq 2\sqrt{eAe \max(\mu + B\nu)}. \tag{74}$$

That completes the proof of Proposition 9. □

Remark 3 If the diffusion matrix field A is not assumed to be uniform in the space variables anymore (but still satisfies (6)), and if $\max \nu > 0$, then the hypothesis for conservation is satisfied for B large enough and the same arguments as above imply that

$$\limsup_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{\max(eAe)}\sqrt{\max \nu}.$$

Proposition 10 *Assume now that* $f_u(x,0) = \mu(x) + B\nu(x)$, *where* $\int_C \mu \geq 0$, $\int_C \nu \geq 0$ *and* $\max \nu > 0$. *Then*

$$2\sqrt{\frac{eAe}{|C|}\int_C \left(\frac{\mu}{B}+\nu\right)dx} \leq \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe\max\left(\frac{\mu}{B}+\nu\right)} \tag{75}$$

and

$$\frac{1}{2}\sqrt{eAe\max\nu} \leq \liminf_{B\to+\infty}\frac{c^*(B)}{\sqrt{B}} \leq \liminf_{B\to+\infty}\frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe\max\nu}. \tag{76}$$

PROOF. As already underlined, the assumptions of Proposition 10 guarantee that the hypothesis for conservation is satisfied for all $B > 0$.

We will now use the characterization of c^* given by (71). Let $\phi_{B,\lambda}$ be defined by (72). Dividing (72) by $\lambda\phi_{B,\lambda}|C|$ and integrating by parts leads to

$$\lambda eAe + \frac{\int_C(\mu+B\nu)}{\lambda|C|} \leq -\frac{k_\lambda(B)}{\lambda}. \tag{77}$$

One deduces from (77) and (71) that

$$2\sqrt{\frac{eAe}{|C|}\int_C(\mu+B\nu)} \leq c^*(B), \tag{78}$$

and the result (75) follows from (74) and (78).

The proof of the lower bound in (76) is divided in two steps. Let

$$0 \leq \gamma := \liminf_{B\to+\infty}\frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe\max\nu}$$

and $(B_n)_{n\in\mathbb{N}} \to +\infty$ such that $c^*(B_n)/\sqrt{B_n} \to \gamma$ as $n \to +\infty$. First, from (71), there exits a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ such that $\frac{-k_{\lambda_n}(B)}{\lambda_n\sqrt{B_n}} \to \gamma$ as $n \to +\infty$. Moreover, from (74), one knows that

$$\frac{-k_{\lambda_n}(B)}{\lambda_n\sqrt{B_n}} \leq 2\sqrt{eAe\max\nu} + \varepsilon_n, \tag{79}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$. Using the equation (77), one obtains with (79)

$$\lambda_n eAe + \frac{\int_C(\mu+B\nu)}{|C|\lambda_n} \leq \frac{-k_{\lambda_n}(B_n)}{\lambda_n} \leq 2\sqrt{B_n eAe\max\nu} + \varepsilon_n\sqrt{B_n}.$$

Assuming that $\int_C \mu \geq 0$ and $\int_C \nu \geq 0$, one deduces that

$$\lambda_n \leq 2\sqrt{\frac{B_n}{eAe}\max\nu} + \varepsilon_n\sqrt{B_n}. \tag{80}$$

Next, consider the eigenvalue problem

$$\begin{cases} -\nabla\cdot(A\nabla\psi_{B,\lambda})-2\lambda Ae\cdot\nabla\psi_{B,\lambda} \\ \qquad -\lambda^2 eAe\psi_{B,\lambda}-(\mu+B\nu)\psi_{B,\lambda}=\widetilde{k}_\lambda(B)\psi_{B,\lambda}, \\ \psi_{B,\lambda}>0 \text{ on } C,\ \psi_{B,\lambda}=0 \text{ on } \partial C,\ \|\psi_{B,\lambda}\|_\infty=1, \end{cases}$$

and let us prove that $\widetilde{k}_\lambda(B)>k_\lambda(B)$ (for all $\lambda>0$ and $B>0$). Assume that, on the contrary, one has $\widetilde{k}_\lambda(B)\le k_\lambda(B)$; then the function $\psi_{B,\lambda}$ satisfies

$$\begin{aligned} -\nabla\cdot(A\nabla\psi_{B,\lambda})-2\lambda Ae\cdot\nabla\psi_{B,\lambda}-\lambda^2 eAe\psi_{B,\lambda} \quad &-(\mu+B\nu)\psi_{B,\lambda}-k_\lambda(B)\psi_{B,\lambda} \\ &=(\widetilde{k}_\lambda(B)-k_\lambda(B))\psi_{B,\lambda}\le 0. \end{aligned} \tag{81}$$

Since the function $\phi_{B,\lambda}$ defined by (72) is positive in $\overline{C}$, one can assume that $\kappa\psi_{B,\lambda}<\phi_{B,\lambda}$ in $\overline{C}$ for all $\kappa>0$ small enough. Now, set

$$\kappa^*=\sup\left\{\kappa>0,\ \kappa\psi_{B,\lambda}<\phi_{B,\lambda} \text{ in } \overline{C}\right\}>0.$$

Then, by continuity, $\kappa^*\psi_{B,\lambda}\le\phi_{B,\lambda}$ in $\overline{C}$ and there exists x_1 in $\overline{C}$ such that $\kappa^*\psi_{B,\lambda}(x_1)=\phi_{B,\lambda}(x_1)$. But, since $\phi_{B,\lambda}>0$ in $\overline{C}$ and $\psi_{B,\lambda}=0$ on ∂C, it follows that $x_1\in C$. Therefore, using (81), it follows from the strong elliptic maximum principle that $\kappa^*\psi_{B,\lambda}\equiv\phi_{B,\lambda}$ in $\overline{C}$, which is impossible from the boundary conditions on ∂C. Finally, one concludes that $\widetilde{k}_\lambda(B)>k_\lambda(B)$.

Let us now define $\Psi_{B,\lambda}(x)=e^{\lambda x\cdot e}\psi_{B,\lambda}(x)$. From (81), the function $\Psi_{B,\lambda}$ satisfies the eigenvalue problem

$$\begin{cases} -\nabla\cdot(A\nabla\Psi_{B,\lambda})-(\mu+B\nu)\Psi_{B,\lambda}=\widetilde{k}_\lambda(B)\Psi_{B,\lambda}, \\ \Psi_{B,\lambda}>0 \text{ on } C,\ \Psi_{B,\lambda}=0 \text{ on } \partial C, \end{cases}$$

and it follows that

$$\widetilde{k}_\lambda(B)=\min_{\psi\in H_0^1(C),\ \psi\not\equiv 0}\frac{\displaystyle\int_C\nabla\psi\cdot(A\nabla\psi)-(\mu(x)+B\nu(x))\psi^2}{\displaystyle\int_C\psi^2}.$$

Let $\varepsilon>0$ be arbitrarily chosen. Then, there exists ψ_ε in $H_0^1(C)$, such that $\|\psi_\varepsilon\|_\infty=1$, $\psi_\varepsilon\ge 0$ and, for all $x\in C$,

$$\psi_\varepsilon(x)>0\Rightarrow(\max\nu\ -\nu(x)<\varepsilon).$$

One then easily obtains (see the proof of Proposition 5.2 in [17])

$$\frac{\displaystyle -\int_C\nabla\psi_\varepsilon\cdot(A(x)\nabla\psi_\varepsilon)+\int_C\mu(x)\psi_\varepsilon^2}{\displaystyle\int_C\psi_\varepsilon^2}+B\left(\max\nu-\varepsilon\right)\le-\widetilde{k}_\lambda(B) \tag{82}$$

for all $\lambda>0$.

Hence, using (80) and (82), and since $\widetilde{k}_{\lambda_n}(B_n)>k_{\lambda_n}(B_n)$ one has

$$\liminf_{n\to+\infty}\frac{-k_{\lambda_n}(B_n)}{\lambda_n\sqrt{B_n}}\ge\frac{1}{2}\sqrt{eAe}\left(\sqrt{\max\nu}-\frac{\varepsilon}{\sqrt{\max\nu}}\right).$$

since $-k_{\lambda_n}(B)\ge 0$. Since $\epsilon>0$ was arbitrary, one concludes that

$$\gamma=\liminf_{B\to+\infty}\frac{c^*(B)}{\sqrt{B}}\ge\frac{1}{2}\sqrt{eAe\max\nu}.$$

The formula (76) follows. □

PROOF of Corollary 1. In the special case where $f_u(x,0) = \mu(x)$ ($\nu = 0$ and, say, $B = 1$) with

$$\int_C \mu(x)dx \geq \mu_0|C| > 0,$$

it follows from the lower bound in (75) that $c^*[\mu] \geq c^*[\mu_0] = 2\sqrt{eAe\ \mu_0}$. □

In other words, an heterogeneous medium increases the biological invasion's speed, in comparison with a constant medium, when $\displaystyle\int_C f_u(x,0)dx > 0$.

Coming back to the case where $f_u(x,0) = \mu(x) + B\nu(x)$, it follows from the Proposition 10 that, even if μ and ν have zero average, it suffices for ν to be positive somewhere for the speed $c^*(B)$ to increase like to the square root of the amplitude of the effective birth rate.

Proposition 11 *Assume that $\mu \equiv 0$, $f_u(x,0) = B\nu(x)$ with $\displaystyle\int_C \nu \geq 0$ and $\max \nu > 0$. Then, one has*

$$\lim_{B\to 0^+} \frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx}.$$

PROOF. First, it follows from (75) that

$$2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx} \leq \frac{c^*(B)}{\sqrt{B}} \tag{83}$$

for all $B > 0$.

In order to establish the opposite inequality at the limit $B \to 0^+$, one shall consider two cases :

Case 1 : $\displaystyle\int_C \nu > 0$. Let $\phi_{B,\lambda}$ be defined by (72) and call $\phi_B = \phi_{B,\lambda_B}$ with

$$\lambda_B = \sqrt{\frac{B}{eAe|C|}\int_C \nu(x)dx}.$$

Multiply (72) by ϕ_B and integrate it by parts over C. Dividing by $\displaystyle\int_C \phi_B^2$, one obtains

$$k_{\lambda_B}(B) = \frac{\displaystyle\int_C \nabla\phi_B A \nabla\phi_B}{\displaystyle\int_C \phi_B^2} - \lambda_B^2 eAe - B\frac{\displaystyle\int_C \nu\phi_B^2}{\displaystyle\int_C \phi_B^2},$$

and

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq \lambda_B eAe + \frac{B}{\lambda_B}\frac{\displaystyle\int_C \nu\phi_B^2}{\displaystyle\int_C \phi_B^2}.$$

Moreover, observing that $\lambda_B \to 0$ as $B \to +\infty$ and arguing as in the proof of Lemma 7, one knows that ϕ_B converges in $C^{2,\beta}(\mathbb{R}^N)$ (for all $0 \leq \beta < 1$) to $\phi_0 \equiv 1$ as $B \to 0$. Therefore, one can write

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq \lambda_B eAe + \frac{B}{\lambda_B |C|} \int_C \nu \ + \frac{B}{\lambda_B} \varepsilon_B, \tag{84}$$

where $\varepsilon_B \to 0$ as $B \to 0$. Replacing λ_B by its value in (84), one obtains

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq 2\sqrt{eAe \frac{B}{|C|} \int \nu} + \sqrt{\frac{BeAe|C|}{\displaystyle\int_C \nu}} \ \varepsilon_B.$$

From the characterization (71) of $c^*(B)$, one then obtains

$$\frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{\frac{eAe}{|C|} \int_C \nu} + \widetilde{\varepsilon}_B,$$

where $\widetilde{\varepsilon}_B \to 0$ as $B \to 0$. Using (83), one concludes that

$$\lim_{B \to 0} \frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|} \int_C \nu}.$$

Case 2 : $\displaystyle\int_C \nu = 0$. Choose now $\lambda_B = \sqrt{\delta B}$ for arbitrary $\delta > 0$. The above arguments imply that

$$\limsup_{B \to 0^+} \frac{c^*(B)}{\sqrt{B}} \leq \sqrt{\delta} \ eAe$$

and the conclusion follows. □

Finally, Theorem 2 follows from the last four propositions.

Partie II : Study of the premixed flame model with heat losses

Sommaire

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1 Introduction and main results

Zeldovitch has shown in [95], using asymptotic methods, that the reaction-diffusion system modelling the propagation of a premixed laminar flame with heat losses has got two travelling wave solutions in the case of high activation energies. More recently, Glangetas and Roquejoffre in [47] have demonstrated the same result, as a consequence of the dispersion relation which has been obtained by Joulin and Clavin ([60],[61]), and proved rigorously in [47]. The case of a more general nonlinearity has been studied by Giovangigli in [46], where the author proves the existence of a solution for a fixed reaction speed c (considering in his work that the minimum heat loss rate parameter λ is an unknown of the problem), with a Lewis Number Le equal to 1. In this paper, we follow the framework of Berestycki, Nicolaenko and Scheurer [23], keeping c as a solution of the problem, and we prove the existence of two solutions for small values of λ. Moreover, the existence result is established for every positive Lewis Number, hence one also solves a non-uniqueness problem. Indeed, in the adiabatic case, Marion [72] has proved the uniqueness of flames when the Lewis number ($1/\Lambda$ in this paper) is greater than 1, and Bonnet has shown in [26] that when the Lewis number is less than unity, uniqueness cannot be generally assumed. Here, we prove that uniqueness never holds in the non-adiabatic case (for small heat losses λ). Besides, we establishe that, in some cases, the solution with the greatest speed converges to the solution of the adiabatic problem as $\lambda \to 0$, whereas the other speed converges to 0.

Moreover, we also compute some new bounds for the solutions. In particular, Giovangigli has proved in [46] that, when $Le = 1$, the reaction speed c was bounded from above by the reaction speed c_{ad} of the adiabatic problem. Here, we extend his result to $Le \leq 1$ (which is physically meaningful since e.g. $Le = 0.4$ for hydrogen), by showing that c is inferior to the reaction speed of an adiabatic scalar problem. Moreover, for all $Le > 0$, we give an explicit upper bound for c, which does not depend on the heat loss rate parameter λ. We also give a lower bound for the unburnt gases after the reaction.

Let Λ and λ be two positive real numbers. The aim of this work is to prove existence and nonexistence results for the following problem :

Finding two nonnegative classical functions u and v and a nonnegative real number c which satisfy

$$\begin{cases} -u'' + cu' &= f(u,v) - \lambda h(u) \\ -\Lambda v'' + cv' &= -f(u,v) \end{cases} \quad \text{on } \mathbb{R}, \tag{1}$$

with the boundary conditions

$$\begin{cases} u(-\infty) = 0, & u(+\infty) = 0, \\ v(-\infty) = 1, & v'(+\infty) = 0. \end{cases} \tag{2}$$

The following assumptions will be made on f in the sequel : there exist two functions p and g such that

$$f(u,v) = p(u)g(v) \text{ in } \mathbb{R} \times \mathbb{R}, \tag{3}$$

where the function p is globally Lipschitz continuous on $\mathbb{R}$, nondecreasing and of "ignition" type :

$$\exists\, \theta \in (0,1) \text{ s.t. } p(x) = 0 \text{ for all } x \leq \theta \text{ and } p(x) > 0 \text{ for all } x > \theta, \tag{4}$$

and the function g is in $C^0(\mathbb{R})$, increasing on $\mathbb{R}_+$ and such that

$$g < 0 \text{ on } \mathbb{R}_-^* \text{ and } \ g(0) = 0; \tag{5}$$

moreover, for all $\gamma > 0$, let us set

$$k^*(\gamma) := \max\left\{ \sup_{s\in(0,1)} \frac{g(\gamma s)}{g(s)}, \gamma \right\},$$

and

$$k_*(\gamma) := \min\left\{ \inf_{s\in(0,1)} \frac{g(\gamma s)}{g(s)}, \gamma \right\},$$

one then assumes that

$$\text{for all } \gamma > 0,\ 0 < k_*(\gamma) \le k^*(\gamma) < +\infty. \tag{6}$$

Hypothesis (6) is for instance satisfied by functions g of the type $g(y) = y^n$, with $n > 0$. It also works with $C^\infty(\mathbb{R})$ functions g such that it exists $n \ge 1$, $n \in \mathbb{N}$ such that $g^{(n)}(0) \neq 0$, where $g^{(n)}$ is the n^{th} derivative of g.

The function h is supposed to be in $C^1(\mathbb{R})$, strictly increasing. Moreover, it satisfies

$$h(0) = 0,\ h(1) = 1, \exists\ \alpha,\ \beta\ \in\ \mathbb{R} \text{ s.t. } 0 < \alpha \le h' \le \beta. \tag{7}$$

One call (u_{ad}, v_{ad}, c_{ad}) the solutions of the following problem without heat loss (see [23] for the existence of such solutions) :

$$\left\{ \begin{array}{rl} -u''_{ad} + c_{ad}u'_{ad} & = f(u_{ad}, v_{ad}) \\ -\Lambda v''_{ad} + c_{ad}v'_{ad} & = -f(u_{ad}, v_{ad}) \end{array} \right. \text{ on } \mathbb{R}, \tag{8}$$

with the boundary conditions

$$\left\{ \begin{array}{ll} u_{ad}(-\infty) = 0, & u_{ad}(+\infty) = 1, \\ v_{ad}(-\infty) = 1, & v_{ad}(+\infty) = 0. \end{array} \right. \tag{9}$$

Remark 1 The reaction term used here is more general than this which was used in [23], nevertheless, the results of [23] can be easily adapted to our case.

Remark 2 In the case $g(y) = y$ and $\Lambda \le 1$ the solution (u_{ad}, v_{ad}, c_{ad}) of (8-9) is shown to be unique (up to translation) in [72]. Moreover, [26] proves that uniqueness does not hold in the general case for $\Lambda > 1$.

Let (u_s, c_s) be the unique solution (see [23]) of the following adiabatic problem

$$-\Lambda u''_s + c_s u'_s = f(u_s, 1 - u_s), \tag{10}$$

with the boundary conditions

$$u_s(-\infty) = 0,\ u_s(+\infty) = 1. \tag{11}$$

Remark 3 Notice that in the case $\Lambda = 1$, the problem (8-9) reduces to the scalar case (10-11), and therefore, uniqueness always holds.

Under this hypothesis, one has the following results :

Theorem 1 *1) For all $\Lambda > 0$, if λ is sufficiently small, there exist two distinct and nontrivial solutions (u_1, v_1, c_1) and (u_2, v_2, c_2) of the problem (1 -2), with $c_1 < c_2$. Moreover, v_i $(i = 1, 2)$ is nonincreasing and, setting $\underline{\Lambda} = \min\{1, \Lambda\}$, one has the following estimate on $v_i(+\infty)$: $v_i(+\infty) < \dfrac{1-\theta}{\underline{\Lambda}}$, $(i = 1, 2)$.*

2) For all $\Lambda > 0$, $c_1 \to 0$ as $\lambda \to 0$. Moreover, if one assumes that $\Lambda \leq 1$ and $g(y) = y$ on $\mathbb{R}$, (u_2, v_2, c_2) converges locally uniformly to (u_{ad}, v_{ad}, c_{ad}), the unique solution of (8-9), as $\lambda \to 0$. The same result holds in the case $\Lambda = 1$, for the more general reaction term $f(x, y)$.

The next Theorem gives new bounds for every solution (u, v, c) of (1-2).

Theorem 2 *1) If $\lambda > \dfrac{f(1,1)}{h(\theta)}$, the problem (1-2) has got no solution.*

2) Assume that g is globally Lipschitz continuous, where K is the Lipschitz constant, then for all nontrivial solution (u, v, c) of (1-2), one has

$$v(+\infty) > exp\left(-\frac{Kp(1)}{\lambda h(\theta)}\right).$$

3) Let (u, v, c) be a solution of (1-2), then, for all $\Lambda \geq 1$, $0 < c \leq c_s$, where (u_s, c_s) is the solution of (10-11).

4) Let (u, v, c) be a solution of (1-2), and set

$$\sigma_1 = \max_{s\in(\theta,1)} \frac{f(s, 1-s)}{s} \text{ and } \sigma_2 = \max_{s\in[0,1]} f(1 - \underline{\Lambda}s, s), \text{ then}$$

$$0 < c < 2\sqrt{\sigma_1 \Lambda} \text{ for all } \Lambda \geq 1 \text{and } 0 < c < \sqrt{\frac{\sigma_2}{\theta}} \text{ for all } \Lambda > 0.$$

2 Proof of the existence theorem

2.1 Equivalence with a problem on $\mathbb{R}_+$

In this section, we will recall some results of Giovangigli. He establishes in [46], Proposition 2.2, that, under conditions (4) and (7),

Proposition 1 *For all $c \geq 0$ and $\lambda \geq 0$, every solution (u,v) of (1-2), after a shift of the origin, is a solution of*

$$\begin{cases} -u'' + cu' &= f(u, v) - \lambda h(u) \\ -\Lambda v'' + cv' &= -f(u, v) \end{cases} \quad on\ (0, +\infty), \tag{12}$$

with the boundary conditions

$$\begin{cases} u(0) = \theta, \quad u'(0) = c\theta + \lambda \int_{-\infty}^{0} h(u_-), \ v'(0) = \frac{c}{\Lambda}(v(0) - 1), \\ u(+\infty) = 0, \quad v'(+\infty) = 0. \end{cases} \tag{13}$$

where u_- is the unique (see [46]) solution of

$$\begin{cases} -u_-'' + cu_-' + \lambda h(u_-) = 0 \quad on\ \mathbb{R}_-, \\ u_-(-\infty) = 0, \ u_-(0) = \theta. \end{cases} \tag{14}$$

Conversely, every solution (u, v) of (12-14) can be extended to $\mathbb{R}$ in such a way that it is a nontrivial solution of (1-2).

Remark 4 This proposition clearly uses the fact that p is of ignition type. It was demonstrated in [46] with $g(y) = y^n$ and a Lewis number $1/\Lambda$ equal to 1, but it is straightforwardly still valid with our more general reaction term $f(x, y)$ and with $1/\Lambda \neq 1$.

Finally, Proposition (1) shows that solving (1-2) is equivalent to finding a solution (u, v, c) of (1) on $\mathbb{R}_+$ with

$$\left\{\begin{array}{lll} u'(0) = cu(0) + k(\theta, c, \lambda), & u(+\infty) = 0, & u(0) = \theta, \\ \Lambda v'(0) = c(v(0) - 1), & v'(+\infty) = 0, & \end{array}\right. \tag{15}$$

where one has set $k(\theta, c, \lambda) = \lambda \int_{-\infty}^{0} h(u_-)$.

2.2 Existence of solutions in bounded domains

In order to be able to use a topological degree argument, we will study the system (1) on a bounded interval of $\mathbb{R}_+$. Namely, for each $a > 0$, one sets

$$I_a = (0, a)\,,$$

and one seeks solutions (u, v, c) in $C^2(I_a, [0, 1])^2 \times \mathbb{R}_+$ of

$$\left\{\begin{array}{rl} -u'' + cu' & = f(u, v) - \lambda h(u) \\ -\Lambda v'' + cv' & = -f(u, v) \end{array}\right. \quad \text{on } I_a, \tag{16}$$

with the boundary conditions

$$\left\{\begin{array}{lll} u'(0) = cu(0) + k(\theta, c, \lambda), & u(0) = \theta, & u(a) = 0 \\ \Lambda v'(0) = c(v(0) - 1), & & v(a) = 0. \end{array}\right. \tag{17}$$

Let us define the Banach space

$$X_a = C^1(I_a, [0, 1]) \times C^1(I_a, [0, 1]) \times \mathbb{R},$$

with the norm $\|(u, v, c)\|_{X_a} = \|u\|_{C^1(I_a)} + \|v\|_{C^1(I_a)} + |c|$, and let J_τ be the application defined by

$$J_\tau : X_a \longrightarrow X_a,\ (u, v, c) \mapsto (U, V, u(0) + c - \theta),$$

where (U, V) is the unique solution of the linear problem

$$\left\{\begin{array}{rl} -U'' + cU' & = \tau[f(u, v) - \lambda h(u)] + (1 - \tau)(v - \lambda u) \\ -\Lambda_\tau V'' + cV' & = -\tau f(u, v) - (1 - \tau)v, \end{array}\right. \tag{18}$$

on $I_a = (0, a)$ with the boundary conditions,

$$\left\{\begin{array}{ll} U'(0) = cU(0) + \tau k(\theta, c, \lambda), & U(a) = 0 \\ \Lambda_\tau V'(0) = cV(0) - c, & V(a) = 0, \end{array}\right. \tag{19}$$

where $\Lambda_\tau = \tau\Lambda + (1 - \tau)$.

Remark 5 This mapping J_τ is close to the one which was used in [46]. One cannot use here the homotopic transformation of [23]. Indeed, we will see in the proof of Proposition 3 that the right-hand side plays a crucial role in the definition of a positive real number c_* such that no solution (u, v, c) exists with $c \in (\varepsilon, c_* - \varepsilon)$ for λ small enough (and ε arbitrarily small).

Let (u_τ, v_τ, c_τ) be a fixed point of the application J_τ, with $\tau \in [0,1]$. In order to compute a topological degree, we will need some *a priori* estimates on (u,v,c). More precisely, we are going to prove the following proposition :

Proposition 2 *Let (u_τ, v_τ, c_τ) be a fixed point of the application J_τ, with $c \geq 0$. Then, it satisfies the following estimates :*

$$\begin{cases} 0 < u_\tau < 1, & 0 < v_\tau < 1, \quad v_\tau \leq \dfrac{1-u_\tau}{\min\{1, \Lambda_\tau\}}, \\ -c_\tau < u'_\tau < c_\tau \left(1 + \dfrac{1}{\Lambda_\tau}\right), & -\dfrac{c_\tau}{\Lambda_\tau} < v'_\tau < 0, \quad 0 < c_\tau \leq \sqrt{\dfrac{f(1,1)+1}{\theta}}. \end{cases} \tag{20}$$

PROOF : In order to improve the readability of the following calculations, we will skip the index τ, replacing (u_τ, v_τ, c_τ) by (u,v,c).

Lemma 1 *One has $v > 0$ on I_a.*

PROOF. For each $\tau \in (0,1)$, v satisfies the following equation

$$-\Lambda_\tau v'' + cv' = -\tau f(u,v) - (1-\tau)v \text{ on } I_a, \tag{21}$$

therefore, one has

$$-\Lambda_\tau v'' + cv' + \gamma v = 0 \text{ on } I_a, \tag{22}$$

where one has set

$$\gamma(x) := \frac{\tau f(u,v) + (1-\tau)v}{v} \text{ if } v(x) \neq 0, \text{ and } \gamma(x) := 1 \text{ if } v(x) = 0.$$

From our hypothesis (3-6), one has $\gamma(x) \geq 0$ on I_a. Therefore, using a result of [45], one can use a maximum principle, which gives us

$$\inf_{x \in I_a} v(x) = \min\{v(0), v(a)\}. \tag{23}$$

Let us assume that $v(0) < 0$, then, from (19), one has $v'(0) \leq 0$. Since $v(a) = 0$, we necessarily have, from (23), $v'(0) = 0$, which implies $c = 0$ from (19). Therefore, from (21), v is concave as long as $v(x) \leq 0$, which is impossible since $v(a) = 0$. It follows that $v \geq 0$. From the strong maximum principle applied to (22), one has $v > 0$ on I_a. □

Lemma 2 *$v < 1$ and $-\dfrac{c}{\Lambda_\tau} < v' < 0$ on I_a.*

PROOF. From (21), and since $v > 0$, one has

$$\left(e^{-\frac{c}{\Lambda_\tau}x} v'\right)' \geq 0 \text{ on } I_a, \tag{24}$$

thus, integrating this equation between $x \in I_a$ and a, one obtains,

$$e^{-\frac{c}{\Lambda_\tau}(a-x)} v'(a) \geq v'(x).$$

But $v > 0$ on I_a and $v(a) = 0$, hence $v'(a) \leq 0$. One deduces $v' \leq 0$. Now, let $x_1 \in I_a$ be such that $v'(x_1) = 0$. Then, integrating (24) between x_1 and x for $x \geq x_1$, one finds $v'(x) \geq 0$. Thus,

v is constant on $[x_1, a]$. This is in contradiction with $v > 0$. Therefore $v' < 0$, and, using (19), $v(0) < 1$, hence $v < 1$ on I_a.

Moreover, from (21), $\Lambda_\tau v' - cv$ is nondecreasing on I_a, thus

$$v'(x) > -\frac{c}{\Lambda_\tau} \text{ on } I_a, \tag{25}$$

from (19), and since $v > 0$. □

Lemma 3 *On has* $0 < u < 1$ *and* $v \leq \dfrac{1-u}{\min\{1, \Lambda_\tau\}}$ *on* I_a.

PROOF. From the strong maximum principle, and using the hypothesis (7), one easily obtains $u > 0$ on I_a.

As it was done in the proof of Lemma 14, set $w = u + v$. Then w satisfies

$$-w'' + cw' = (\Lambda_\tau - 1)v'' - \lambda[\tau h(u) + (1-\tau)u].$$

An integration between 0 and $x \in I_a$ gives, using (19),

$$-w'(x) + cw(x) - c + k \leq (\Lambda_\tau - 1)v'(x) \text{ on } I_a,$$

since $h(u) \geq 0$. Thus, since $k \geq 0$, one has

$$-(we^{-cx})' \leq e^{-cx}(\Lambda_\tau - 1)v'(x) \text{ on } I_a,$$

and integrating between x and a, one obtains

$$w(x) \leq (\Lambda_\tau - 1)e^{cx} \int_x^a v'(t)e^{-ct}dt + 1, \tag{26}$$

thus, if $\Lambda_\tau > 1$, one has $u + v \leq 1$ since $v' \leq 0$ from the above calculations. Next, if $\Lambda_\tau \leq 1$, we deduce from (26) that

$$w \leq (1 - \Lambda_\tau)v + 1, \tag{27}$$

therefore, $u \leq 1 - \Lambda_\tau v$. Thus, since $v > 0$, in both cases, $u < 1$. Besides, one has $v \leq \dfrac{1-u}{\min\{1, \Lambda_\tau\}}$.
□

Lemma 4 $-c < u' < c\left(1 + \frac{1}{\Lambda_\tau}\right)$ *on* I_a.

PROOF. Let us add the equations satisfied by u and v, and integrate the sum between 0 and x. Using the boundary conditions (19), one obtains,

$$u' - cu \geq -c + \tau k + c - \Lambda_\tau v' > -c.$$

It follows that $u' > -c$.

Moreover, setting $y = u + \Lambda_\tau v$, one has

$$-(y'e^{-cx})' \leq e^{-cx}v'c(\Lambda_\tau - 1),$$

and an integration from $x \in I_a$ to a gives

$$y'(x)e^{-cx} \leq c(\Lambda_\tau - 1)\int_x^a e^{-ct}v'(t)dt,$$

since $y'(a) \leq 0$. Thus, since $-\Lambda_\tau v' < c$ (from (25)), one deduces that

$$u'(x)c + c(\Lambda_\tau - 1)\int_x^a e^{c(x-t)}v'(t)dt \text{ on } I_a, \tag{28}$$

therefore, if $\Lambda_\tau \geq 1$, then $u' < c$, and for $\Lambda_\tau < 1$, from (28) and using again $-\Lambda_\tau v' < c$, one has

$$u'(x) < c + \frac{c}{\Lambda_\tau}e^{cx}\int_x^a ce^{-ct}dt < c + \frac{c}{\Lambda_\tau}(1 - e^{c(x-a)}),$$

thus $u' < c\left(1 + \frac{1}{\Lambda_\tau}\right)$. □

Lemma 5 *One has* $0 < c \leq \sqrt{\dfrac{f(1,1)+1}{\theta}}$.

PROOF. One knows that u satisfies,

$$-u'' + cu' = \tau[f(u,v) - \lambda h(u)] + (1-\tau)(v - \lambda u) \text{ on } I_a. \tag{29}$$

Let us multiply (29) by u and integrate it from 0 to a. One obtains

$$\begin{aligned} -u(a)u'(a) + u(0)u'(0) + \int_0^a |u'|^2 + \quad & c/2\left(u(a)^2 - u(0)^2\right) \\ & < \int_0^a \tau u f(u,v) + (1-\tau)uv, \end{aligned} \tag{30}$$

therefore, from (19),

$$\int_0^a |u'|^2 < \theta(\frac{c}{2}\theta + \tau k) + \int_{-a}^a \tau u f(u,v) + (1-\tau)uv, \tag{31}$$

but, since $u < 1$ and from (21), one has,

$$\begin{aligned} \int_0^a \tau u f(u,v) + (1-\tau)uv &\leq \int_0^a \tau f(u,v) + (1-\tau)v, \\ &= \int_0^a \Lambda_\tau v'' - cv', \\ &= \Lambda_\tau v'(a) - \Lambda_\tau v'(0) - cv(a) + cv(0), \\ &\leq c. \end{aligned}$$

Therefore, combining this inequality with (31), we obtain,

$$\int_0^a |u'|^2 < c, \tag{32}$$

therefore, one has $c > 0$.

In order to obtain an upper bound for c, let us consider a function w, which is the solution of the following boundary value problem, with $M := f(1,1) + 1$,

$$\begin{cases} -w'' + cw' = M \text{ on } I_a, \\ w'(0) = cw(0) + \tau k, \\ w(a) = 0. \end{cases} \tag{33}$$

The function w is a super-solution of the equation satisfied by u. Therefore, from the boundary conditions of w, we deduce, with the maximum principle, that $w \geq u$ on I_a. A direct computation gives

$$w(0) = \frac{1}{c^2}\left[M - 4\tau kc + 4\tau kce^{-ca} - M(1+ac)e^{-ca}\right].$$

But, since $w(0) \geq u(0)$ one has $\frac{M}{c^2} - 4\frac{\tau k}{c}(1-e^{-ca}) \geq \theta$ thus

$$\frac{M}{c^2} \geq \theta + 4\frac{\tau k}{c}(1-e^{-ca}) \geq \theta.$$

Finally, $c \leq \sqrt{\frac{M}{\theta}}$. □

Proposition (2) is proved. □

In order to compute a topological degree, we need an other estimate, that will be established for small parameters λ.

Let $(u_\tau, v_\tau, c_\tau) \in X_a$ be a fixed point of J_τ.

Proposition 3 *For all $\varepsilon > 0$, there exist $\lambda_1 > 0$ and $a_1 > 0$ such that for all $a > a_1$, for all $\tau \in [0,1]$,*

$$(\lambda < \lambda_1) \Longrightarrow (c_\tau \notin (\varepsilon, c_* - \varepsilon)),$$

where c_* is a positive real number defined at the end of the proof.

PROOF : Let (u_n, v_n, c_n, τ_n) be a sequence of fixed points of J_{τ_n}, with $c_n \in (\varepsilon, c_* - \varepsilon)$, $\tau_n \in [0,1]$, $a_n \to +\infty$, $\lambda_n \to 0$ and $\varepsilon > 0$. Then, from Proposition (2), the sequence $(u_n, v_n, c_n, \tau_n)_{n\in\mathbb{N}}$ is bounded in $\left[C^2(0,a_n)\right]^2 \times \mathbb{R} \times [0,1]$. By compactness, we obtain the convergence (up to the extraction of some subsequence), in $C^1_{loc}(\mathbb{R}_+)^2 \times \mathbb{R} \times [0,1]$ of (u_n, v_n, c_n, τ_n) to (u, v, c, τ) which is a classical solution of

$$\left\{\begin{array}{ll} -u'' + cu' &= \tau f(u,v) + (1-\tau)v \\ -\Lambda_\tau v'' + cv' &= -\tau f(u,v) - (1-\tau)v \end{array}\right. \quad \text{on } \mathbb{R}_+, \tag{34}$$

with

$$u'(0) = c\theta, \ \Lambda_\tau v'(0) = cv(0) - c, \ u(0) = \theta, \tag{35}$$

since $k(\theta, c, 0) = 0$ (see (19)), and

$$c \in [\varepsilon, c_* - \varepsilon]. \tag{36}$$

Let us set

$$\Lambda_* = \min(1, \frac{1}{\Lambda_\tau}) \text{ and } \Lambda^* = \max(1, \frac{1}{\Lambda_\tau}).$$

One has the following

Lemma 6 $\Lambda_*(1-u) \leq v \leq \Lambda^*(1-u)$.

PROOF : $v \leq \Lambda^*(1-u)$ is a consequence of Proposition (2). Setting $w = u+v$ and $y = u + \Lambda_\tau v$, one has

$$\left\{\begin{array}{l} -w'' + cw' = (\Lambda_\tau - 1)v'' \\ -y'' + cy' = (\Lambda_\tau - 1)cv' \end{array}\right. \quad \text{on } \mathbb{R}_+.$$

Integrating the former expressions from 0 to $x \in \mathbb{R}_+$ and using (35), we obtain

$$\begin{cases} -w'(x) + cw(x) + (1-\Lambda_\tau)v'(x) = c \\ -y'(x) + cy(x) + c(1-\Lambda_\tau)v(x) = c \end{cases} \text{on } \mathbb{R}_+, \tag{37}$$

thus

$$\begin{cases} (w(x)e^{-cx})' = (v'(x)(1-\Lambda_\tau) - c)e^{-cx} \\ (y(x)e^{-cx})' = (v(x)(1-\Lambda_\tau) - 1)ce^{-cx} \end{cases} \text{on } \mathbb{R}_+,$$

hence,

$$\begin{cases} w(x) = 1 + (\Lambda_\tau - 1)e^{cx}\displaystyle\int_x^{+\infty} v'(t)e^{-ct}dt \\ y(x) = 1 + c(\Lambda_\tau - 1)e^{cx}\displaystyle\int_x^{+\infty} v(t)e^{-ct}dt \end{cases} \text{on } \mathbb{R}_+,$$

therefore,

$$\begin{cases} v(x) = 1 - u(x) + (\Lambda_\tau - 1)e^{cx}\displaystyle\int_x^{+\infty} v'(t)e^{-ct}dt \\ v(x) = \dfrac{1-u(x)}{\Lambda_\tau} + \dfrac{(\Lambda_\tau - 1)}{\Lambda_\tau}e^{cx}\displaystyle\int_x^{+\infty} v(t)e^{-ct}dt \end{cases} \text{on } \mathbb{R}_+.$$

Therefore, if $\Lambda_\tau \le 1$, then $v \ge 1-u$ since $v' \le 0$, and if $\Lambda > 1$, then $v \ge \dfrac{1-u}{\Lambda_\tau}$ since $v > 0$. That concludes the proof of the lemma. □

Lemma 7 *One has* $u'(+\infty) = v'(+\infty) = u''(+\infty) = v''(+\infty) = 0$.

PROOF : Let us integrate the equation satisfied by v from 0 to $x \in \mathbb{R}_+$. One has

$$\Lambda_\tau v'(x) - cv(x) = -c + \int_0^x [\tau f(u,v) + (1-\tau)v(x)]\,dx. \tag{38}$$

Since v is nonincreasing and nonnegative, it has got a finite limit as $x \to +\infty$. Moreover, since $\tau f(u,v) + (1-\tau)v(x) \ge 0$, the integral $\displaystyle\int_0^x [\tau f(u,v) + (1-\tau)v(x)]\,dx$ converges (even to $+\infty$). Thus, from (38) $v'(x)$ has a limit as $x \to +\infty$. Since v is bounded, this limit is equal to 0. Moreover,

Lemma 8 *The function* $u + v - 1$ *has a constant sign on* $\mathbb{R}_+$.

PROOF : From Lemma 6, $(1-u)(\Lambda_* - 1) \le u+v-1 \le (1-u)(\Lambda^* - 1)$. Thus, if $\Lambda_\tau > 1$, $\Lambda^* = 1$ and $u + v - 1 \le 0$, and if $\Lambda_\tau \le 1$ $\Lambda_* = 1$ and one has $u + v - 1 \ge 0$. □

Using the equation satisfied by y in (37), we obtain

$$\Lambda_\tau v'(x) + u'(x) = c(u(x) + v(x) - 1). \tag{39}$$

Therefore, $\Lambda_\tau v(x) + u(x) = c\int_0^x (u(t) + v(t) - 1)dt + \Lambda_\tau v(0) + \theta$. But, from Lemma 8 the integral of the right-hand side converges, hence u admits a limit at $+\infty$. Arguing as for v we deduce that $u'(+\infty) = 0$. It then follows from (34) that $u''(+\infty) = v''(+\infty) = 0$. That completes the proof of Lemma 7 . □

Lemma 9 *$u' \geq 0$ on $\mathbb{R}_+$*

PROOF : The equation satisfied by u in (34) is equivalent to

$$-(u'(x)e^{-cx})' = e^{-cx}\left[\tau f(u(x), v(x)) + (1-\tau)v(x)\right] \geq 0.$$

Therefore, integrating this expression from $x \in \mathbb{R}_+$ to $+\infty$, and using Lemma 7 one obtains the sought result. □

Lemma 10 *$u(+\infty) = 1$ and $v(+\infty) = 0$.*

PROOF : From (34) and (39), $u(+\infty)$ and $v(+\infty)$ satisfy

$$\begin{cases} \tau f[u(+\infty), v(+\infty)] + v(+\infty)(1-\tau) = 0 \\ u(+\infty) + v(+\infty) = 1. \end{cases}$$

Thus, since one has $c > \varepsilon$, one has $u'(0) = c\theta > 0$. Therefore, from Lemma 9, $u(+\infty) > \theta$. It follows that $v(+\infty) = 0$ and $u(+\infty) = 1$. □

Using Lemma 6, we obtain :

$$\begin{aligned} -u'' + cu' &= \tau f(u, v) + (1-\tau)v && (40) \\ &\leq \tau f(u, \Lambda^*(1-u)) + (1-\tau)\Lambda^*(1-u). && (41) \end{aligned}$$

Moreover, from the hypothesis (6) and from the definition of $k^*(\Lambda^*)$, one has $f(u, \Lambda^*(1-u)) \leq k^*(\Lambda^*) f(u, 1-u)$, and

$$\Lambda^* \leq k^*(\Lambda^*) < +\infty, \tag{42}$$

hence, from (41) and (42),

$$-u'' + cu' \leq k^*(\Lambda^*)[\tau f(u, 1-u) + (1-\tau)(1-u)]. \tag{43}$$

Let us now multiply the inequality (43) by u and integrate over $(0, +\infty)$. One then obtain, using again Lemma 7,

$$\int_0^\infty (u')^2 + \frac{c}{2}(1+\theta^2) \leq k^*(\Lambda^*) \int_0^\infty [\tau f(u, 1-u) + (1-\tau)(1-u)], \tag{44}$$

since $u \leq 1$. Next, still using Lemma 6, one obtains

$$\begin{aligned} -u'' + cu' &= \tau f(u, v) + (1-\tau)v \\ &\geq \tau f(u, \Lambda_*(1-u)) + (1-\tau)\Lambda_*(1-u). && (45) \end{aligned}$$

Again, from the definition of $k_*(\Lambda_*)$, one has $f(u, \Lambda_*(1-u)) \geq k_*(\Lambda_*) f(u, 1-u)$ and

$$0 < k_*(\Lambda_*) \leq \Lambda_*. \tag{46}$$

Then, integrating (45) over $(0, +\infty)$, and using (46), one has, from the limiting behaviors obtained in the previous lemmas,

$$c \geq k_*(\Lambda_*) \int_0^\infty [\tau f(u, 1-u) + (1-\tau)(1-u)],$$

hence, with (44), one obtains the inequality

$$\int_0^\infty (u')^2 + \frac{c}{2}(1+\theta^2) \leq \frac{k^*(\Lambda^*)}{k_*(\Lambda_*)}c,$$

which is equivalent to

$$\int_0^\infty (u')^2 \leq \frac{c}{2}\left(2\frac{k^*(\Lambda^*)}{k_*(\Lambda_*)} - 1 - \theta^2\right). \tag{47}$$

Next, let us multiply the equality (40) by u' and integrate it over $(0,+\infty)$. Using again the lemma 6 and the result (46) above, one gets

$$\frac{\theta^2c^2}{2} + c\int_0^\infty (u')^2 \geq k_*(\Lambda_*)\int_0^\infty u'[\tau f(u,1-u) + (1-\tau)(1-u)],$$

which gives, using Lemma 9 and inequality (47),

$$\begin{aligned}\frac{c^2}{2}\left(2\frac{k^*(\Lambda^*)}{k_*(\Lambda_*)} - 1\right) &\geq k_*(\Lambda_*)\int_\theta^1 [\tau f(s,1-s) + (1-\tau)(1-s)]ds \\ &\geq k_*(\Lambda_*)m_*,\end{aligned}$$

where m_* is defined by $m_* = \min(m_1, m_2)$, with $m_1 := \displaystyle\int_\theta^1 f(s,1-s)ds$, and $m_2 = \displaystyle\int_\theta^1 (1-s)ds$.

Therefore, one has

$$c^2 \geq 2\frac{k_*(\Lambda_*)^2}{2k^*(\Lambda^*) - k_*(\Lambda_*)}m_* > 0 \text{ from (42) and (46)}, \tag{48}$$

thus, if $\Lambda < 1$ then $\Lambda_* = 1$ and $k_*(\Lambda_*) = 1$. Furthermore, $\Lambda^* \leq \dfrac{1}{\Lambda}$, which implies from the definition of k^* that $k^*(\Lambda^*) \leq k^*\left(\dfrac{1}{\Lambda}\right)$. Similarly, for $\Lambda \geq 1$, $\Lambda^* = 1$ and $k^*(\Lambda^*) = 1$. Moreover, $\Lambda_* \geq \dfrac{1}{\Lambda}$, which implies that $k_*(\Lambda_*) \geq k_*\left(\dfrac{1}{\Lambda}\right)$. Besides, it follows from (42) and (46) that $\dfrac{1}{\Lambda} \leq k^*\left(\dfrac{1}{\Lambda}\right) < +\infty$ and $0 < k_*\left(\dfrac{1}{\Lambda}\right) \leq \dfrac{1}{\Lambda}$. Hence one can set

$$c_* := \sqrt{\frac{2}{2k^*\left(\frac{1}{\Lambda}\right) - 1}m_*} \text{ if } \Lambda < 1,$$

and,

$$c_* := \sqrt{\frac{2k_*\left(\frac{1}{\Lambda}\right)^2}{2 - k_*\left(\frac{1}{\Lambda}\right)}m_*} \text{ if } \Lambda \geq 1.$$

Then c_* is positive and independent of τ (although Λ^* and Λ_* were depending on τ). Moreover, one has $c \geq c_*$, which is in contradiction with (36). That completes the proof of Proposition 3. □

In order to compute a topological degree, we need to investigate the case $\tau = 0$.

The 3-uplet (u, v, c) is a fixed point of J_0 in X_a if and only if it satisfies

$$\begin{cases} -u'' + cu' + \lambda u = v \\ -v'' + cv' + v = 0 \end{cases} \text{ in } I_a, \tag{49}$$

with the boundary conditions

$$\left\{\begin{array}{ll} u'(0) = cu(0), & u(a) = 0, \\ v'(0) = cv(0) - c, & v(a) = 0, \end{array}\right. \tag{50}$$

and

$$u(0) = \theta. \tag{51}$$

Proposition 4 *For λ small enough and a large enough, (49-51) admits exactly two solutions (u^1, v^1, c^1) and (u^2, v^2, c^2). Moreover, there exists $c_\theta > 0$, such that for all $\varepsilon > 0$, $\exists\ \lambda_2 > 0$ and $a_2 > 0$, such that for all $\lambda < \lambda_2$ and for all $a > a_2$, one has $0 < c^1 < \varepsilon$ and $c^2 > c_\theta$.*

PROOF. For each c there is only one couple of function (u, v) which satisfies (49) and (50). One can compute this functions explicitly. Let us set $\phi(c) := u(0)(c)$. In order to solve the equation $\phi(c) = \theta$, we study the function $c \mapsto \phi(c)$ on $\mathbb{R}_+$. The results of Appendix show that for λ small enough and a large enough, the equation $\phi(c) = \theta$ admits exactly two solutions c^1 and c^2, with $0 < c^1 < (2\lambda)^{1/3}$ and $c^2 > c_\theta$, where c_θ is positive and does not depend on λ and a. The proof of Proposition 4 is then complete. □

Using Propositions 2, 3 and 4, we are now able to define topological degrees. First, set $K_\tau \equiv I - J_\tau$, where I is the identity mapping of X_a. Coming back to the definition of J_τ, with (18) and (19), we can easily check that U and V are in bounded in $C^2(I_a)$, if u and v are in $C^1(I_a)$. Moreover $C^2(I_a)$ is compactly embedded into $C^1(I_a)$. Therefore the mappings J_τ and K_τ are compact. Similarly, the mapping $F : X_a \times [0, 1] \to (X_a \mapsto X_a)$ defined by $(u, v, c, \tau) \mapsto K_\tau$ is compact and uniformly continuous with respect to τ. Moreover, one notices that (u, v, c) is a classical solution of (16-17) if and only if $K_1(u, v, c) = 0$, therefore, we will look for the solutions of this equation. From the properties of K_τ, and using the homotopic invariance property of the topological degree (see [79]), we will only need to compute the degree for $\tau = 0$. At first let us prove that a degree can be defined. Set $\underline{\Lambda} := \min(\Lambda, 1)$, $M := f(1, 1) + 1$ and

$$\Omega = \left\{(u, v, c) \in X_a : \|(u, v, c)\|_{(X_a)} < 2 + 2\sqrt{\frac{M}{\theta}}(1 + \frac{1}{\underline{\Lambda}})\right\}. \tag{52}$$

From Proposition 2, one knows that

Proposition 5 *For all $\tau \in [0, 1]$, and for a large enough,*

$$0 \notin K_\tau(\partial\Omega). \tag{53}$$

Next, we are going to define two open sets of X_a, in which the topological degree will be calculated. Let us set $\varepsilon > 0$ such that $\varepsilon < \min\left\{\frac{c_*}{8}, c_\theta\right\}$ where c_* and c_θ are defined in Propositions 3 and 4 respectively, $a_* := \max\{a_1, a_2\}$ and $\lambda_* := \min\{\lambda_1, \lambda_2\}$, where a_1, a_2, λ_1 and λ_2 are defined in Propositions 3 and 4.

Now, set

$$\begin{array}{l} O_1^a := \Omega \cap \{(u, v, c) \in X_a \ s.t.\ c < 2\varepsilon\}, \text{ and} \\ O_2^a := \Omega \cap \{(u, v, c) \in X_a \ s.t.\ c > c_* - 2\varepsilon\}. \end{array} \tag{54}$$

Notice that $O_2^a \neq \{\emptyset\}$: since $\varepsilon < c_\theta$, one deduces from Propositions 3 and 4 that $c^2 > c_* - \varepsilon$ (c^2 is defined in Proposition 4), thus Proposition 2 ensures that

$$\sqrt{\frac{M}{\theta}} > c^2 > c_* - \varepsilon > c_* - 2\varepsilon. \tag{55}$$

One has the following

Proposition 6 *For all $\lambda < \lambda_*$, for all $\tau \in [0,1]$ one has*

$$0 \ \notin \ K_\tau(\partial O_i^a),$$

for $i = 1, 2$, and for $a \geq a_*$.

PROOF : Let (u_i^*, v_i^*, c_i^*) be a solution of $K_\tau = 0$ in ∂O_i^a $(i = 1, 2)$. Then one deduces from Proposition 5 that $c_1^* = 2\varepsilon$ (and $c_2^* = c_* - 2\varepsilon$). But, from Proposition 3, $c_i^* \notin (\varepsilon, c_* - \varepsilon)$ $(i = 1, 2)$, thus one gets a contradiction, and the proposition is proved. □

Proposition 6 enables us to define the Leray-Schauder degree $deg(K_\tau, O_i^a, 0)$, for $i = 1, 2$. Therefore, one can compute its value :

Proposition 7 *Let us assume that $\lambda < \lambda_*$ and $a \geq a_*$; for $i = 1, 2$, and for all $\tau \in [0,1]$, one has $deg(K_\tau, O_1^a, 0) = deg(K_0, O_1^a, 0) = 1$ and $deg(K_\tau, O_2^a, 0) = deg(K_0, O_2^a, 0) = -1$.*

PROOF. From the compactness of the mapping K_τ, and its uniform continuity with respect to τ, one obtains, using the homotopic invariance of the Leray-Schauder degree (see [79]),

$$deg(K_1, O_i^a, 0) = deg(K_0, O_i^a, 0).$$

Furthermore, K_0 is known explicitly :

$$K_0 : X_a \to X_a : (u, v, c) \mapsto (u - U_0(c), v - V_0(c), u(0) - \theta)$$

where U_0 and V_0 are the solutions of (18-19). Moreover, this mapping is homotopic to

$$\Phi : X_a \to X_a : (u, v, c) \mapsto (u - U_0(c), v - V_0(c), \phi(c) - \theta),$$

where ϕ is defined as in the proof of Proposition 4. Besides, since $\lambda < \lambda_* \leq \lambda_2$, $c^1 < \varepsilon$ and, as it was noticed in (55), $c^2 > c_* - \varepsilon > c_* - 2\varepsilon$ (c^1 and c^2 are defined in Proposition 4). Hence the equation $\phi(c) = \theta$ admits exactly one solution in $(0, 2\varepsilon)$ (resp. in $(c_* - 2\varepsilon, \sqrt{\frac{M}{\theta}})$). Using the multiplicative property of the degree, one finds that $deg(\Phi, O_1^a, 0) = 1$ and $deg(\Phi, O_2^a, 0) = -1$ for $a \geq a_*$ (the sign of the degree is given by the sign of ϕ', see [23]). □

Remark 6 The real ε can be chosen as small as one wants, provided λ is sufficiently small.

As a consequence of Proposition 7, it follows that :

Corollary 1 *For $\lambda < \lambda_*$, and for $a > a_*$, the problem (16-17) admits at least two classical solutions (u_1^a, v_1^a, c_1^a) and (u_2^a, v_2^a, c_2^a), with $0 < c_1^a < \frac{c_*}{4}$ and $\frac{3c_*}{4} < c_2^a$. Moreover, from Remark 6, one can assume that $c_1^a < r_\lambda$, where r_λ does not depend on a and $r_\lambda \to 0$ as $\lambda \to 0$.*

2.3 Passage to the limit in $\mathbb{R}_+$

For $\lambda < \lambda_*$, and for a large enough, let (u_1^a, v_1^a, c_1^a) and (u_2^a, v_2^a, c_2^a) be the solutions obtained in Corollary 1. Using Proposition 2, one finds that, for a large enough, (u_1^a, v_1^a, c_1^a) (resp. (u_2^a, v_2^a, c_2^a)) is bounded (independently of a) in $C^2(I_a)\times C^2(I_a)\times\mathbb{R}_+$. By compactness, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $(u_1^{a_n}, v_1^{a_n}, c_1^{a_n})$ (resp.$(u_2^{a_n}, v_2^{a_n}, c_2^{a_n})$) converges in $C^1_{loc}(\mathbb{R}_+)\times C^1_{loc}(\mathbb{R}_+)\times\mathbb{R}_+$ to (u_1, v_1, c_1) (resp. (u_2, v_2, c_2)). One may use the general notation (u, v, c) for (u_1, v_1, c_1) and (u_2, v_2, c_2) in the sequel, when the distinction between the two solutions is not needed. (u, v, c) satisfies :

$$\left\{\begin{array}{l} -u'' + cu' = f(u,v) - \lambda h(u) \\ -\Lambda v'' + cv' = -f(u,v) \end{array}\right. \quad \text{on } \mathbb{R}, \tag{56}$$

with the boundary conditions,

$$u'(0) = cu(0) + k(\theta, c, \lambda),\ u(0) = \theta,\ \Lambda v'(0) = cv(0) - c. \tag{57}$$

Moreover, since the real numbers λ_*, c_* and r_λ of Corollary 1 are independent of a, one has

$$0 \le c_1 \le r_\lambda \text{ and } \frac{3c_*}{4} \le c_2. \tag{58}$$

Now let us prove that

Lemma 11 *The two solutions found above satisfy*

$$u(+\infty) = 0,\ u'(+\infty) = v'(+\infty) = 0 \textit{ and } u''(+\infty) = v''(+\infty) = 0.$$

PROOF : Passing to the limit $a \to +\infty$ in Proposition 2, we find that v is nonnegative and nonincreasing. Let us integrate the equation satisfied by v in (56) from 0 to $x > 0$. One obtains

$$\Lambda v'(x) - cv(x) = \Lambda v'(0) - cv(0) + \int_0^x f(u,v)(s)ds. \tag{59}$$

Therefore, since $f(u, v) \ge 0$ on $\mathbb{R}_+$, the right-hand side of the above equation converges as $x \to +\infty$. Hence $v'(+\infty)$ is defined. Since v is bounded, $v'(+\infty) = 0$.

Next, let us add the equations satisfied by u and v in (56). That gives, after an integration from 0 to $x > 0$:

$$-\Lambda v'(x) - u'(x) + cu(x) + cv(x) = \tag{60}$$

$$-\Lambda v'(0) + u'(0) - cu(0) - cv(0) - \lambda \int_0^x h(u)(s)ds. \tag{61}$$

Since the left-hand side is bounded, and since $h(u)$ is nonnegative (because $u \ge 0$), one has $\lim_{x\to+\infty} h(u)(x) = 0$. From the hypothesis (7) on h, it follows that $u(+\infty) = 0$. Now, using the above results on $v'(+\infty)$, the equation (61) implies that $u'(+\infty)$ exits. Since u is bounded, one finds $u'(+\infty) = 0$. Then $u''(+\infty) = v''(+\infty) = 0$ immediately follows from (56). □

From Proposition 1, we have found two distinct solutions of (1-2). In order to complete the proof of Theorem 1, it only remains to prove that this two solutions are nontrivial. Since $u(0) = \theta$ and $u(+\infty) = 0$, u is not a trivial solution. Let us assume that v is a constant. Then, it follows from (56) that $f(u, v) = 0$ on $\mathbb{R}$, hence $-u'' + cu' + \lambda h(u) = 0$ on $\mathbb{R}$. Since $u(\pm\infty) = 0$, one obtains that $u \equiv 0$ from the maximum principle, which is in contradiction with $u(0) = \theta$.

The upper bound of $v(+\infty)$ in Theorem 1 follows from Proposition 2. Indeed, one has $v(0) \leq \dfrac{1-u(0)}{\min\{1,\Lambda\}}$, and since v is nonincreasing, nontrivial, and $u(0)=\theta$, one has $v(+\infty) < v(0) \leq \dfrac{1-\theta}{\min\{1,\Lambda\}}$. Part 1) of Theorem 1 is then proved.

2.4 Passage to the limit $\lambda \to 0$

In this subsection, one studies the behavior of the two solutions (u_1, v_1, c_1) and (u_2, v_2, c_2) found above. Let us recall that, besides satisfying (1-2), (u_i, v_i, c_i) verifies

$$u_i'(0) = c_i u_i(0) + k(\theta, c_i, \lambda),\ u_i(0) = \theta,\ \Lambda v_i'(0) = c_i v_i(0) - c_i, \tag{62}$$

with

$$0 \leq c_1 \leq r_\lambda \text{ and } \frac{3c_*}{4} \leq c_2, \tag{63}$$

and

$$\|u\|_{C^1(I_a)} \leq 1 + \sqrt{\frac{M}{\theta}}(1 + \frac{1}{\underline{\Lambda}}),\ \|v\|_{C^1(I_a)} \leq 1 + \frac{\sqrt{M}}{\sqrt{\theta}\underline{\Lambda}}, \tag{64}$$

for $i = 1, 2$.

Let $\lambda \to 0$. Since (u_i, v_i, c_i) is bounded independently of λ in $C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times \mathbb{R}_+$ from (1) and (64), by compactness it follows that (u_i, v_i, c_i) converges, up to the extraction of some subsequence, in $C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times \mathbb{R}_+$ to a solution (u_i^0, v_i^0, c_i^0) of

$$\begin{cases} -(u_i^0)'' + c_i^0 (u_i^0)' &= f(u_i^0, v_i^0) \\ -\Lambda (v_i^0)'' + c_i^0 (v_i^0)' &= -f(u_i^0, v_i^0) \end{cases} \text{ on } \mathbb{R}, \tag{65}$$

with

$$(u_i^0)'(0) = c_i^0 \theta,\ \Lambda (v_i^0)'(0) = c_i^0 v_i^0(0) - c_i^0,\ u_i^0(0) = \theta, \tag{66}$$

since $k(\theta, c, 0) = 0$ (see (19)), and

$$c_1^0 = 0,\ \frac{3c_*}{4} \leq c_2^0, \tag{67}$$

since $\lim_{\lambda \to 0} r_\lambda = 0$. Therefore, since $0 < \dfrac{3c_*}{4} \leq c_2^0$, one can apply Lemma (10) for (u_2^0, v_2^0, c_2^0), which gives

$$u_2^0(+\infty) = 1 \text{ and } v_2^0(+\infty) = 0. \tag{68}$$

Besides, since for each $\lambda > 0$, $u_2 \leq \theta$ on $\mathbb{R}_-$, one has

$$\begin{cases} -(u_2^0)'' + c_2^0 (u_2^0)' &= 0 \\ -\Lambda (v_2^0)'' + c_2^0 (v_2^0)' &= 0 \end{cases} \text{ on } \mathbb{R}_-. \tag{69}$$

Moreover, u_2 is nondecreasing on $\mathbb{R}_-$ and v_2 is nonincreasing on $\mathbb{R}_-$; it follows that u_2^0 is nondecreasing on $\mathbb{R}_-$ and v_2^0 is nonincreasing on $\mathbb{R}_-$. Therefore, from (66) and (69) one has

$$u_2^0(-\infty) = 0 \text{ and } v_2^0(-\infty) = 1. \tag{70}$$

Hence, (u_2^0, v_2^0, c_2^0) is a solution of (8-9).

Let us study now (u_1^0, v_1^0, c_1^0). As seen in (67), $c_1^0 = 0$. Since $u_1 \geq 0$ and $v_1 \geq 0$ for all λ, one has $u_1^0 \geq 0$ and $v_1^0 \geq 0$. It then follows from (65) that u_1^0 is concave. Since u_1^0 is bounded, it follows that $u_1^0 \equiv u_1^0(0) = \theta$. Similarly, $v_1^0 \equiv K_1$, where K_1 is an unknown constant.

Let us assume now that $\Lambda \leq 1$ and $g(y) = y$ on $\mathbb{R}$. Then, as mentioned in Remark 2 the problem (8-9) has a unique solution. Since one has just demonstrated that for all converging subsequence $(u_2, v_2, c_2)_k$ of (u_2, v_2, c_2), $(u_2, v_2, c_2)_k$ converges to a solution of (8-9), this uniqueness result allows us to say that the whole sequence (u_2, v_2, c_2) converges to the solution of (8-9).

Similarly, it follows from Remark 3 that, when $\Lambda = 1$, for the general reaction term $f(x, y)$, the whole sequence (u_2, v_2, c_2) converges to the solution of (8-9).

That concludes the proof of of Theorem 1, part 2).

3 Proof of theorem 2

In this section, we give the proof of Theorem 2, establishing some results about the general solutions of (1-2).

3.1 An upper bound for λ

Let us prove that problem (1-2) has no solution for λ large enough.

Lemma 12 *Let (u, v, c) be a nontrivial solution of (1-2) with $u \geq 0$ and $v \geq 0$. Then $c > 0$.*

PROOF : Assume by contradiction that $c = 0$. Then, from (1), one has

$$\Lambda v'' = f(u, v) \geq 0 \text{ on } \mathbb{R}, \tag{71}$$

thus v is a convex function. From (2), it follows that the function v is constant. Hence, (71) gives $f(u, v) = 0$, thus, from (1), one has $-u'' = -\lambda h(u) \leq 0$. Similarly, it follows from the boundary conditions (2) that u is a constant function. The solution (u, v, c) is then trivial. □

Lemma 13 *Let (u, v, c) be a solution of (1-2) with $u \geq 0$ and $v \geq 0$. Then v is nonincreasing and $v \leq 1$.*

PROOF : The equation satisfied by v is equivalent to

$$\left(v' e^{\frac{-c}{\Lambda} x}\right)' = \frac{f(u, v)}{\Lambda} \text{ on } \mathbb{R}.$$

Integrating this equation between x and $+\infty$, one obtains, using (2) and $v \geq 0$,

$$v' e^{\frac{-c}{\Lambda} x} \leq 0 \text{ on } \mathbb{R}, \tag{72}$$

therefore, v is nonincreasing, and from (2) one has $v \leq 1$. □

Lemma 14 *Under the assumptions of Lemma 13, one has $u \leq 1$.*

PROOF : Set $w = u + v$ then w satisfies the following equation : $-w'' + cw' = (\Lambda - 1)v'' - \lambda h(u)$. Integrating it between $-\infty$ and $x \in \mathbb{R}$, one obtains, using (2),

$$-w'(x) + cw(x) - c \leq (\Lambda - 1)v'(x) \text{ on } \mathbb{R},$$

since $h(u) \geq 0$. Therefore, integrating between x and $+\infty$, one has

$$w(x) \leq (\Lambda - 1)e^{cx} \int_x^{+\infty} v'(t)e^{-ct}dt + 1, \tag{73}$$

then, if $\Lambda > 1$, one has $u + v \leq 1$ since $v' \leq 0$ from Lemma 13. Next, if $\Lambda \leq 1$, we deduce from (73) that

$$w \leq (1 - \Lambda)v + 1, \tag{74}$$

thus, $u \leq 1 - \Lambda v$. Finally, in both cases, $u \leq 1$, and the lemma is proved. □

Now, let us integrate the equation satisfied by u between $-\infty$ and $+\infty$; we obtain, from (2),

$$\int_{-\infty}^{+\infty} f(u,v) - \lambda \int_{-\infty}^{+\infty} h(u) = 0. \tag{75}$$

(Indeed, $u'(\pm\infty) = 0$, see for instance Lemma 11). Moreover, from Lemmas 13 and 14, one has $0 \leq v \leq 1$ and $\leq u \leq 1$, therefore, using the hypothesis (3-7), one obtains that

$$f(u,v) \leq f(1,1)\frac{h(u)}{h(\theta)}.$$

Therefore, from (75), we deduce that

$$\int_{-\infty}^{+\infty} f(1,1)\frac{h(u)}{h(\theta)} \geq \lambda \int_{-\infty}^{+\infty} h(u)$$

thus

$$\lambda \leq \frac{f(1,1)}{h(\theta)}.$$

That completes the proof of Theorem 2, part 1).

3.2 A lower bound for the unburnt gases

In order to establish this lower bound, we need some more computations. Assume that the function g is Lipschitz-continuous on $\mathbb{R}$, and let (u, v, c) be a nontrivial solution of (1-2).

Let us show at first that $v(+\infty) \neq 0$. Since the solution (u, v, c) is nontrivial, one can assume that it satisfies $u(0) = \theta$ (see Proposition 1). Thus one can define x_0, as the smallest $y \in [0, +\infty)$ such that $u(y) = \theta$ and $u \leq \theta$ for all $x \geq y$. Hence (u, v, c) satisfies the following problem

$$\left\{ \begin{array}{rl} -u'' + cu' + \lambda h(u) & = 0 \\ -\Lambda v'' + cv' & = 0 \end{array} \right. \text{ on } (x_0, +\infty), \tag{76}$$

with the boundary conditions

$$\left\{ \begin{array}{ll} u(x_0) = \theta, & u(+\infty) = 0, \\ v'(+\infty) = 0. & \end{array} \right. \tag{77}$$

It immediately follows that $v \equiv v(+\infty)$ on $(x_0, +\infty)$. Moreover, using Proposition 1 (14), one knows that $u'(x_0) = c\theta - \lambda \int_{x_0}^{+\infty} h[u_+(s)]ds$, where u_+ is the unique solution of

$$\left\{ \begin{array}{ll} -u_+'' + cu_+' + \lambda h(u_+) = 0 & \text{on } (x_0, +\infty), \\ u_+(x_0) = \theta, \ u_+(+\infty) = 0. & \end{array} \right. \tag{78}$$

Besides, if $v(+\infty) = 0$, one has

$$v(x_0) = 0,\ v'(x_0) = 0,\ u(x_0) = \theta \text{ and } u'(x_0) = c\theta - \lambda \int_{x_0}^{+\infty} h[u_+(s)]ds.$$

It follows from the Cauchy-Lipschitz uniqueness theorem that $v \equiv 0$, hence a contradiction. Therefore, one has shown that $v_\infty := v(+\infty) > 0$.

Now, dividing by v the equation satisfied by v in (1), and integrating by parts over $\mathbb{R}$, one obtains, using $v_\infty > 0$, $v' \leq 0$ (see Lemma 13) and (2),

$$-\Lambda \int_{\mathbb{R}} \left[\frac{v'}{v}\right]^2 + cln(v_\infty) = -\int_{\mathbb{R}} \frac{f(u,v)}{v}, \tag{79}$$

and it follows that

$$cln(v_\infty) \geq -\int_{\mathbb{R}} \frac{f(u,v)}{v}. \tag{80}$$

Let K be the Lipschitz constant of g. One has $\frac{g(v)}{v} \leq K$ on $\mathbb{R}$, thus, since $f(u,v) = p(u)g(v)$, one obtains

$$cln(v_\infty) \geq -K \int_{\mathbb{R}} p(u). \tag{81}$$

Since $p(u) \leq h(u)\frac{p(1)}{h(\theta)}$, it follows from (81) that

$$cln(v_\infty) \geq -K\frac{p(1)}{h(\theta)} \int_{\mathbb{R}} h(u), \tag{82}$$

and adding the equations in (1) and integrating over $\mathbb{R}$ one obtains

$$\int_{\mathbb{R}} h(u) = \frac{c}{\lambda}(1 - v_\infty). \tag{83}$$

From (82) and (83), one deduces that

$$ln(v_\infty) \geq -K\frac{p(1)}{\lambda h(\theta)}(1 - v_\infty) > -\frac{Kp(1)}{\lambda h(\theta)}. \tag{84}$$

Finally one deduces that

$$v_\infty > exp\left(-\frac{Kp(1)}{\lambda h(\theta)}\right),$$

and part 2) of Theorem 2 is proved.

3.3 Upper bounds for the speed c

Let x_0 be defined as in the subsection 3.2. One can again assume (up to translation) that every solution (u, v, c) of (1-2) satisfies $u(0) = \theta$ and $u \leq \theta$ on $(-\infty, 0)$.

3.3.1 Comparison with an adiabatic problem

In this subsection, one assumes that $\Lambda \geq 1$. As it was proved in [23], one knows that the following problem admits a unique solution (u_s, c_s) :

$$-\Lambda u_s'' + c_s u_s' = f(u_s, 1 - u_s), \tag{85}$$

with the boundary conditions

$$u_s(-\infty) = 0,\ u_s(+\infty) = 1 \text{ and } u_s(0) = \theta. \tag{86}$$

Moreover, u_s is strictly increasing, therefore, setting $w = 1 - u_s$, one can define a function k by $k(y) := -w' \circ w^{-1}(1-y)$, moreover, $k \in C^1(0,1)$, $k > 0$ on $(0,1)$ and $k(\theta) = u_s'(0) = \dfrac{c_s\theta}{\Lambda}$.

Then one need the

Lemma 15 *Let (u, v, c) be a solution of (1-2) with $c > 0$, $u \geq 0$ and $v \geq 0$. Then v is decreasing on $(-\infty, x_0)$.*

PROOF : It is similar to that of Lemma 13, using $v'(x_0) = 0$. □

Let us set $j(y) := -v' \circ v^{-1}(1-y)$. Then, from Lemma 15, j is well defined and $j \in C^1([1 - v(0), 1 - v_\infty])$. Moreover, $j > 0$ on $(1 - v(0), 1 - v_\infty)$, and

$$j(1 - v(0)) = \frac{c}{\Lambda}(1 - v(0)),\ j(1 - v_\infty) = v'(x_0) = 0. \tag{87}$$

The equation satisfied by v in 1 verifies

$$-\Lambda v''(v^{-1}(1-y)) + cv'(v^{-1}(1-y)) = -f(u(v^{-1}(1-y)), 1-y),$$

for y in $(1 - v(0), 1 - v_\infty)$. But, since $\Lambda \geq 1$, one knows from the proof of Lemma 14 that $u \leq 1 - v$ thus

$$-\Lambda v''(v^{-1}(1-y)) + cv'(v^{-1}(1-y)) \geq -f(1 - v \circ v^{-1}(1-y), 1-y),$$

for y in $(1 - v(0), 1 - v_\infty)$, which finally gives

$$(\Lambda j j' - cj)(y) \geq -f(y, 1-y) \text{ in } (1 - v(0), 1 - v_\infty). \tag{88}$$

Similarly, one has

$$(\Lambda k k' - c_s k)(y) = -f(y, 1-y) \text{ in } (0,1). \tag{89}$$

Moreover, since $k > 0$, one deduces from (89) that $\left(k(y) - \dfrac{c_s y}{\Lambda}\right)' < 0$, therefore integrating between θ and $1 - v(0)$, one gets

$$k(1 - v(0)) < \frac{c_s}{\Lambda}(1 - v(0)) = \frac{c_s}{c} j(1 - v(0)),$$

by (87),thus

$$\frac{k(1 - v(0))}{j(1 - v(0))} < \frac{c_s}{c}. \tag{90}$$

Then, subtracting the equations (88) and (89), one obtains,

$$\frac{\Lambda}{2}\left(j^2 - k^2\right)' \geq c\left(j - \frac{c_s}{c}k\right) \text{ in } (1 - v(0), 1 - v_\infty). \tag{91}$$

Now, assume that $\frac{c_s}{c} < 1$, then, from (90) and (91),

$$\left(j^2 - k^2\right)' (1 - v(0)) > 0, and \tag{92}$$

$$\left(j^2 - k^2\right) (1 - v(0)) > 0. \tag{93}$$

Let us assume now that the set $\{y \in (1 - v(0), 1 - v_\infty) \text{ s.t. } (j^2 - k^2)(y) = 0\}$ is nonempty and admits a lower bound y_1. Then $(j^2 - k^2)(y_1) = 0$ and, from (91), it follows that $(j^2 - k^2)'(y_1) > 0$ which is impossible from the definition of y_1. Therefore, $j^2 > k^2$ on $(1 - v(0), 1 - v_\infty)$, which is impossible since (87) gives $j(1 - v_\infty) = 0$, and $k > 0$ in $(0, 1)$. Finally, one obtains

$$\frac{c_s}{c} \geq 1, \tag{94}$$

and Theorem 2, part 3), is proved.

3.3.2 Computation of explicit upper bounds for c

Let us assume that $\Lambda \geq 1$, and set

$$\sigma_1 = \max_{s \in (\theta, 1)} \frac{f(s, 1 - s)}{s}, \tag{95}$$

one obtains, using (89),

$$k' \geq \frac{c_s}{\Lambda} - \frac{\sigma_1 y}{\Lambda k(y)} \text{ for } y \in (0, 1). \tag{96}$$

Let us assume that $c_s > 2\sqrt{\sigma_1 \Lambda}$. Now, as done in [72] on bounded intervals, one can set $m(y) = ry$, with $r = \frac{c_s + \sqrt{c_s^2 - 4\sigma_1 \Lambda}}{2\Lambda}$. Then one has the

Lemma 16 *$k(y) > m(y)$ for all $y \in (\theta, 1)$.*

PROOF : First, let us notice that

$$m'(y) = \frac{c_s}{\Lambda} - \frac{\sigma_1 y}{\Lambda m(y)} \text{ for } y \in (0, 1). \tag{97}$$

Therefore, since $m(\theta) = r\theta < \frac{c_s}{\Lambda}\theta$, and $k(\theta) = -w'(0) = \frac{c_s}{\Lambda}\theta$, one has $k(\theta) > m(\theta)$. Moreover, from (96) and (97),

$$(k - m)'(y) \geq \frac{\sigma_1 y}{\Lambda} \left(\frac{1}{m(y)} - \frac{1}{k(y)} \right) \text{ for } y \in (\theta, 1).$$

It follows that $k(y) > m(y)$ on $(\theta, 1)$. $\square$

Hence, for all $x \in (0, +\infty)$, $m(1 - w(x)) < k(1 - w(x))$ thus $r(1 - w(x)) < -w'(x)$ which is equivalent to

$$r < \frac{(u_s)'(x)}{u_s(x)} \text{ for } x \in (0, x_0), \tag{98}$$

where u_s is defined by (85-86). Integrating (98) between 0 and $a > 0$, one obtains

$$\ln\left(\frac{u_s(a)}{\theta}\right) > ra. \tag{99}$$

It follows from the definition of r that, for a chosen large enough, c_s can be as small as one wants, which is in contradiction with the hypothesis $c_s \geq 2\sqrt{\sigma_1 \Lambda}$. Finally, it follows that $c_s < 2\sqrt{\sigma_1 \Lambda}$, and from (94), one has

$$c < 2\sqrt{\sigma_1 \Lambda}.$$

Let us now compute another upper bound for c, which depends on θ, but works for all $\Lambda \geq 0$. Let us set $\sigma_2 = \max\limits_{s\in[0,1]} f(1-\underline{\Lambda}s, s)$, and let w be the solution of

$$\begin{cases} -w'' + cw' = \sigma_2 \text{ on } (0,a), \\ w'(0) = cw(0) + k(\theta, c, \lambda), \\ w(a) = 1, \end{cases} \tag{100}$$

where $k(\theta, c, \lambda)$ is defined as in (15). The function w can be computed explicitly :

$$w(x) = X_1(a) + X_2(a)e^{cx} + \frac{\sigma_2}{c}x, \tag{101}$$

where X_1, X_2 are two real valued functions.

Moreover, the equation $w'(0) = cw(0) + k(\theta, c, \lambda)$ gives $X_1(a) = \dfrac{\sigma_2}{c^2} - \dfrac{k}{c}$, and $\lim\limits_{a\to+\infty} X_2(a) = 0$ since $w(a) = 1$. Finally, one gets $w(0) = X_1(a) + X_2(a) \to \dfrac{\sigma_2}{c^2} - \dfrac{k}{c}$ as $a \to +\infty$.

Since the function w is a super-solution of the equation satisfied by u, one deduces from the boundary conditions of w together with the maximum principle, that $w \geq u$ on $(0,a)$, hence $w(0) = X_1(a) + X_2(a) \geq u(0) = \theta$. Therefore, taking a large enough, one deduces that

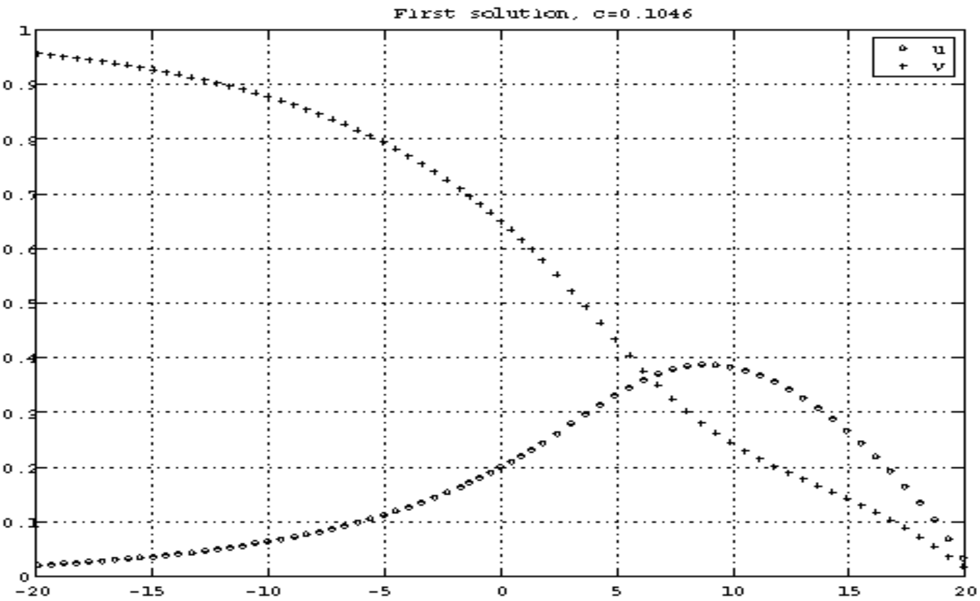
$$c < \sqrt{\frac{\sigma_2}{\theta}}.$$

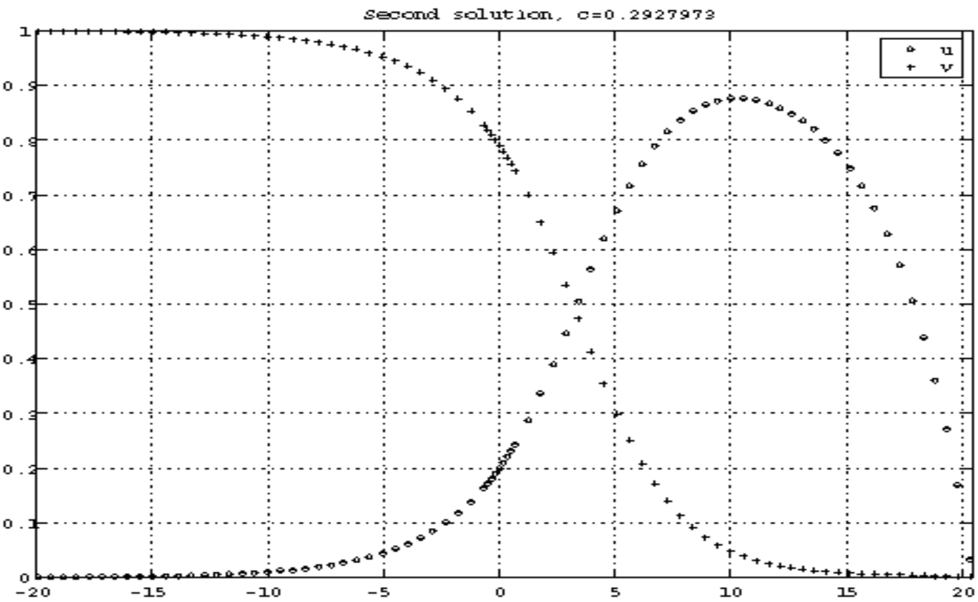
The proof of Theorem 2 is complete.

4 Numerical results

In this section we give some numerical approximations of the two solutions obtained in Corollary 1, on a bounded domain $[0, a]$. Numerically, no other solution has been found. A shooting method on c and $v(0)$ has been used to compute these approximations, and the solutions have been extended to $[-a, a]$.

We have taken here $f(u,v) = vH(u-\theta)(u-\theta)^2$, where H is the Heaviside function, and $h(u) = u$. The values of the parameters are : $\Lambda = 1$, $\lambda = 0.001$, $\theta = 0.2$ and $a = 20$.





One can notice that the second solution is close to the adiabatic one.

Appendix

Let us recall the notations of Proposition 4 :

$$\begin{aligned}\phi(c) &= \frac{c}{\lambda-1}\Big(\frac{1}{e^{a\Delta\lambda}r_2+r_2-c}+\frac{1}{e^{-a\Delta\lambda}r_1+r_1-c}\\ &- \frac{1}{e^{a\Delta}z_2+z_2-c}-\frac{1}{e^{-a\Delta}z_1+z_1-c}\Big),\end{aligned}$$

and $\Delta_\lambda := \sqrt{c^2+4\lambda}$, $\Delta := \sqrt{c^2+4}$, $r_1 = \dfrac{c-\Delta_\lambda}{2}$, $r_2 = \dfrac{c+\Delta_\lambda}{2}$, $z_1 = \dfrac{c-\Delta}{2}$, and $z_2 = \dfrac{c+\Delta}{2}$.
It is straightforward that ϕ is a $C^2([0,+\infty))$ function. Set $\phi' := \dfrac{\partial}{\partial c}\phi$.

Proposition 8 *For $\lambda < 1$ and a large enough, ϕ' vanishes only once on $(0,+\infty)$.*

PROOF : Let us show at first that one has

Lemma 17 *Let μ be the function such that, for a large enough and for all c in $(0,+\infty)$, $\phi(c) = \frac{2}{1-\lambda}(\frac{c}{c+\Delta_\lambda} - \frac{c}{c+\Delta}) + \mu(c,a)$. Then the function $\mu_a(c) := c \mapsto \mu(c,a)$ is in $C^2([0,+\infty))$ for a large enough and*

$$|\,(\mu(c,a))^{(n)}\,| \leq \frac{4}{1-\lambda}e^{\frac{-a\Delta_\lambda}{2}} \text{ for all } n \in \{0,1,2\},$$

where $(\mu(c,a))^{(n)}$ is the n^{th} derivative of the function μ with respect to the variable c.

PROOF : Setting $\mu(c,a) := \phi(c) - \frac{2}{1-\lambda}(\frac{c}{c+\Delta_\lambda} - \frac{c}{c+\Delta})$, it easily follows that $c \mapsto \mu(c,a)$ is a $C^2([0,+\infty))$ function for a large enough. Moreover, one can notice that

$$\left|\left(\frac{c}{e^{a\Delta_\lambda}r_2 + r_2 - c}\right)^{(n)}\right| \leq e^{-\frac{a\Delta_\lambda}{2}}, \text{ for all } n \in \{0,1,2\}.$$

One similarly has

$$\left|\left(\frac{c}{e^{a\Delta}z_2 + z_2 - c}\right)^{(n)}\right| \leq e^{-\frac{a\Delta_\lambda}{2}}, \text{ for all } n \in \{0,1,2\}.$$

Furthermore, one can also easily check that

$$\left|\left(\frac{2c}{c+\Delta_\lambda} + \frac{c}{e^{-a\Delta_\lambda}r_1 + r_1 - c}\right)^{(n)}\right| \leq e^{-\frac{a\Delta_\lambda}{2}}, \text{ for all } n \in \{0,1,2\} \text{ and}$$

$$\left|\left(\frac{2c}{c+\Delta} + \frac{c}{e^{-a\Delta}z_1 + z_1 - c}\right)^{(n)}\right| \leq e^{-\frac{a\Delta_\lambda}{2}}, \text{ for all } n \in \{0,1,2\}.$$

Finally, the lemma 17 follows from the above calculations.□

In the sequel, one writes $\mu'(c,a)$ for $\dfrac{\partial\mu}{\partial c}(c,a)$, and $\mu''(c,a)$ for $\dfrac{\partial\mu}{\partial c^2}(c,a)$.
Let us set $b(c) := \dfrac{c}{c+\Delta_\lambda} - \dfrac{c}{c+\Delta}$.

Lemma 18 *The first derivative of b vanishes only once on $(0,+\infty)$.*

PROOF : Let us notice at first that $\Delta'_\lambda = \dfrac{c}{\Delta_\lambda}$, $\Delta' = \dfrac{c}{\Delta}$, $\Delta''_\lambda = \dfrac{1}{\Delta_\lambda} - \dfrac{c^2}{\Delta_\lambda^3}$ and $\Delta'' = \dfrac{1}{\Delta} - \dfrac{c^2}{\Delta^3}$.
Then one has

$$b'(c) = \frac{c+\Delta_\lambda - c(1+\Delta'_\lambda)}{(c+\Delta_\lambda)^2} - \frac{c+\Delta - c(1+\Delta')}{(c+\Delta)^2} = \frac{\Delta_\lambda - \frac{c^2}{\Delta_\lambda}}{(c+\Delta_\lambda)^2} - \frac{\Delta - \frac{c^2}{\Delta}}{(c+\Delta)^2}.$$

Setting $k(c) = b'(c)(c+\Delta_\lambda)^2(c+\Delta)^2$, one obtains

$$k(c) = \left(\Delta_\lambda - \frac{c^2}{\Delta_\lambda}\right)(c+\Delta)^2 - \left(\Delta - \frac{c^2}{\Delta}\right)(c+\Delta_\lambda)^2,$$

thus one has $k'(c) = 2(c+\Delta)(1+\frac{c}{\Delta})\left(\Delta_\lambda - \frac{c^2}{\Delta_\lambda}\right) - 2(c+\Delta_\lambda)(1+\frac{c}{\Delta_\lambda})\left(\Delta - \frac{c^2}{\Delta}\right)$ $+c\left[\left(-\frac{1}{\Delta_\lambda} + \frac{c^2}{\Delta_\lambda^3}\right)(c+\Delta)^2 + \left(\frac{1}{\Delta_\lambda} - \frac{c^2}{\Delta_\lambda^3}\right)(c+\Delta_\lambda)^2\right]$. Hence, $k'(c) = 5c^3\left(\frac{1}{\Delta} - \frac{1}{\Delta_\lambda}\right) + 4c\,(\Delta_\lambda - \Delta) + 6c^2\left(\frac{\Delta_\lambda}{\Delta} - \frac{\Delta}{\Delta_\lambda}\right) + c^5\left(\frac{1}{\Delta^3} - \frac{1}{\Delta_\lambda^3}\right) + 2c^4\left(\frac{\Delta}{\Delta_\lambda^3} - \frac{\Delta_\lambda}{\Delta^3}\right) + c^3\left(\frac{\Delta^2}{\Delta_\lambda^3} - \frac{\Delta_{\lambda^2}}{\Delta^3}\right) + c\left(\frac{\Delta_\lambda^2}{\Delta} - \frac{\Delta^2}{\Delta_\lambda}\right)$. Therefore, with obvious notations, one can write $k'(c) = (t1) + (t2) + (t3) + (t4) + (t5) + (t6) + (t7)$.

Moreover,

$$\begin{aligned}
(t2) &= -4c\Delta\Delta_\lambda\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right),\\
(t3) &= 6c^2\Delta\Delta_\lambda\left(\frac{1}{\Delta^2} - \frac{1}{\Delta_\lambda^2}\right),\\
&= -6c^2(\Delta_\lambda + \Delta)\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right),\\
(t4) &= c^5\left(\frac{1}{\Delta_\lambda^2} + \frac{1}{\Delta\Delta_\lambda} + \frac{1}{\Delta^2}\right)\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right),
\end{aligned}$$

$$\begin{aligned}
(t5) &= 2c^4\Delta\Delta_\lambda\left(\frac{1}{\Delta_\lambda^4} - \frac{1}{\Delta^4}\right),\\
&= 2c^4\left(\frac{\Delta}{\Delta_\lambda^2} + \frac{1}{\Delta_\lambda} + \frac{1}{\Delta} + \frac{\Delta_\lambda}{\Delta^2}\right)\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right),\\
(t6) &= c^3\Delta^2\Delta_\lambda^2\left(\frac{1}{\Delta_\lambda^5} - \frac{1}{\Delta^5}\right),\\
&= c^3\left(\frac{\Delta^2}{\Delta_\lambda^2} + \frac{\Delta}{\Delta_\lambda} + 1 + \frac{\Delta_\lambda}{\Delta} + \frac{\Delta_{\lambda^2}}{\Delta^2}\right)\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right),\\
(t7) &= c\Delta^2\Delta_\lambda^2\left(\frac{1}{\Delta^3} - \frac{1}{\Delta_\lambda^3}\right),\\
&= -c\left(\Delta_\lambda^2 + \Delta\Delta_\lambda + \Delta^2\right)\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right).
\end{aligned}$$

Therefore, one can write $k'(c) = c\left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta}\right)I(c)$, with

$$\begin{aligned}
I(c) &= -5c^2 - 4\Delta\Delta_\lambda - 6c(\Delta_\lambda + \Delta) + c^4\left(\frac{1}{\Delta_\lambda^2} + \frac{1}{\Delta\Delta_\lambda} + \frac{1}{\Delta^2}\right)\\
&+ 2c^3\left(\frac{\Delta}{\Delta_\lambda^2} + \frac{1}{\Delta_\lambda} + \frac{1}{\Delta} + \frac{\Delta_\lambda}{\Delta^2}\right) + c^2\left(\frac{\Delta^2}{\Delta_\lambda^2} + \frac{\Delta}{\Delta_\lambda} + 1 + \frac{\Delta_\lambda}{\Delta} + \frac{\Delta_{\lambda^2}}{\Delta^2}\right)\\
&- (\Delta_\lambda^2 + \Delta\Delta_\lambda + \Delta^2).
\end{aligned}$$

Moreover, since $c < \Delta_\lambda < \Delta$, one has $I(c) \leq -8c^2$. Finally,

$$k'(c) \leq -8c^3 \left(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta} \right) < 0. \tag{102}$$

Moreover, from a series expansion of $k(c)$ at $c = +\infty$, one obtains

$$\lim_{c \to +\infty} k(c) = 8(\lambda - 1) < 0, \tag{103}$$

and, since

$$k(0) = 8\sqrt{\lambda} - 8\lambda = 8\sqrt{\lambda}\left(1 - \sqrt{\lambda}\right) > 0, \tag{104}$$

one deduces from (102-104) that k vanishes only once, and since $k(c) = b'(c)(c+\Delta_\lambda)^2(c+\Delta)^2$, the lemma 18 is proved. $\square$

Let us now prove Proposition 8.
One has $\phi'(c) = \frac{2}{1-\lambda} b'(c) + \mu'(c, a)$, thus, from a straightforward computation,

$$\phi'(0) = \frac{1 - \sqrt{\lambda}}{\sqrt{\lambda}\,(1 - \lambda)} + \mu'(0, a).$$

Therefore, one deduces from Lemma 17 that for a large enough,

$$\phi'(0) > \frac{1}{2} \frac{1 - \sqrt{\lambda}}{\sqrt{\lambda}\,(1 - \lambda)} > 0. \tag{105}$$

Besides, it follows from (103) that

$$b'(c) - \frac{\lambda - 1}{2c^2} \to 0 \text{ as } c \to +\infty. \tag{106}$$

Thus it follows from Lemma 17 that $\phi'(c) < -\dfrac{1}{2c^2} < 0$ for c large enough. Hence, by continuity, ϕ' vanishes at least once on $(0, +\infty)$.

From (104) and (106), one knows that $b'(0) > 0$, and $b'(c) < 0$ for c large enough. Therefore, it follows from Lemma 18 that there exists a unique $c_0 > 0$ such that $b' > 0$ on $[0, c_0)$ $b'(c_0) = 0$ et $b' < 0$ on $(c_0, +\infty)$. Let c_1 and c_2, be two points in $\mathbb{R}_+$ such that $\phi'(c_1) = \phi'(c_2) = 0$. Then

$$|b'(c_1)| \leq \frac{1 - \lambda}{2} |\mu'(c_1, a)|. \tag{107}$$

Using Lemma 18, and from the continuity of b', $|c_0 - c_1| \to 0$ as $b'(c_*) \to 0$. Hence Lemma 17 and (107) give

$$|c_0 - c_1| \to 0 \text{ as } a \to +\infty. \tag{108}$$

Assume that $c_1 \neq c_2$ and set $l(c) := \dfrac{1-\lambda}{2} \phi'(c)(c+\Delta_\lambda)^2(c+\Delta)^2$, then $l(c_1) = l(c_2) = 0$ and there exist $c_3 \in [c_1, c_2]$ such that $l'(c_3) = 0$, since $c_1 \neq c_2$. Thus, since

$$l(c) = k(c) + \frac{1 - \lambda}{2}(c + \Delta_\lambda)^2(c + \Delta)^2 \mu'(c, a),$$

one has $k'(c_3) = t_1(c_3)\mu'(c_3, a) + t_2(c_3)\mu''(c_3, a)$, where t_1 and t_2 are two functions with polynomial increasing.

From (108), one can take a large enough such that $|c_0 - c_1| < \frac{c_0}{2}$, and since $c_0 > 0$, $c_1 > \frac{c_0}{2}$. Thus, since (102) gives $k'(c) \leq -8c^3(\frac{1}{\Delta_\lambda} - \frac{1}{\Delta})$, one has

$$k'(c_3) \leq \max_{c > \frac{c_0}{2}} \left\{ -8c^3 \left(\frac{1}{\Delta_\lambda(c)} - \frac{1}{\Delta(c)} \right) \right\} \leq -c_0^3 \left(\frac{1}{\Delta_\lambda(\frac{c_0}{2})} - \frac{1}{\Delta(\frac{c_0}{2})} \right) < 0. \tag{109}$$

But, from Lemma 17, one can assume for a large enough that

$$\left|k'(c_3)\right| = \left|t_1(c_3)\mu'(c_3, a) + t_2(c_3)\mu''(c_3, a)\right| < \frac{c_0^3}{2} \left(\frac{1}{\Delta_\lambda(\frac{c_0}{2})} - \frac{1}{\Delta(\frac{c_0}{2})} \right),$$

therefore, (109) gives $c_0^3 \left(\frac{1}{\Delta_\lambda(\frac{c_0}{2})} - \frac{1}{\Delta(\frac{c_0}{2})} \right) < \frac{c_0^3}{2} \left(\frac{1}{\Delta_\lambda(\frac{c_0}{2})} - \frac{1}{\Delta(\frac{c_0}{2})} \right)$. Hence one gets a contradiction, and one concludes that $c_1 = c_2$. Proposition 8 is then proved. □

Proposition 9 *There exists $c_\theta > 0$ such that for λ small enough and a large enough, the equation $\phi(c) = \theta$ admits exactly two solutions c^1 and c^2, with $0 < c^1 < (2\lambda)^{1/3}$ and $c^2 > c_\theta$.*

PROOF : Let us begin with the lemma

Lemma 19 *The function ϕ admits a unique maximum.*

PROOF : One has seen in (105) that $\phi'(0) > 0$ for a large enough. Moreover, $\phi(0) = 0$ and $\phi(+\infty) = 0$. The proof of Lemma 19 then follows from Proposition 8.□

Let us now compute $\phi\left[(2\lambda)^{1/3}\right]$. A series expansion at $\lambda = 0$ gives

$$\phi\left[(2\lambda)^{1/3}\right] = 1 - \frac{3}{2^{2/3}}\lambda^{1/3} + \mu(\sqrt{\lambda}, a).$$

Therefore, for a large enough and λ small enough, $\phi\left[(2\lambda)^{1/3}\right]$ can be as close as one wants to 1. It then follows from Lemma 19, with $\phi(0) = 0$, $\phi(+\infty) = 0$ and $\theta < 1$, that the equation $\phi(c) = \theta$ admits exactly two solutions c^1 and c^2, with $0 < c^1 < (2\lambda)^{1/3}$.

Next, a series expansion of $\phi(c)$ at $\lambda = 0$ gives

$$\phi(c) = \frac{\sqrt{c^2+4} - c}{\sqrt{c^2+4} + c} + O(\lambda) + \mu(\sqrt{\lambda}, a),$$

and, with another series expansion at $c = 0$, one obtains

$$\phi(c) = 1 - c + O(c^2) + O(\lambda) + \mu(\sqrt{\lambda}, a),$$

thus, since $\theta < 1$, there exists $c_\theta > 0$ independent of λ and a, such that for λ small enough and a large enough, $\phi(c_\theta) > \theta$. It then follows from Lemma 19 that $c^2 > c_\theta$. Proposition 9 follows. □

Partie III : Une première approche de la modélisation en biologie

Sommaire

Ce travail s'intègre dans le cadre du projet "Caractérisation des potentialités invasives d'un nouveau ravageur des graines de cèdre récemment introduit au Mont-Ventoux" conclu entre l'INRA et le Ministère de l'Agriculture (convention DERF-INRA Num. 61.45.19/01).

1 Introduction

L'insecte dont nous allons modéliser la dispersion, un ravageur au nom de *Megastigmus schimitscheki* Novitzky, a vraisemblablement été introduit depuis le Proche-Orient avec des graines de cèdre du Liban (*Cedrus libani* A. Rich.).

Dans une situation marquée au plan international par la multiplication des invasions biologiques, l'étude de ce processus invasif présente de nombreux intérêts, tant au plan fondamental qu'appliqué. Il s'agit d'une invasion récente, de dispersion encore très limitée, les études menées au Mont-Ventoux ([37], [38], [40]) ayant permis de dater avec précision l'année d'introduction à 1994. Cela facilite l'étude des processus de dispersion spatio-temporelle à partir du point source, au contraire d'espèces plus anciennes et maintenant réparties sur l'ensemble du territoire.

L'étude précédente effectuée dans le cadre de ce projet ([40]) avait pour but, entre autres, d'estimer l'impact de *M. schimitscheki* sur la régénération naturelle des forêts de cèdre, et de comparer les caractéristiques biologiques des populations de la zone d'invasion avec celles observées dans leur milieu naturel (Asie Mineure). Elle nous fournit ainsi des données pour la construction du modèle. Par ailleurs, si le processus d'invasion et d'installation de *M. schimitscheki* n'était que peu documenté, sa biologie a été relativement bien étudiée dans les régions d'origine ([50], [63], [81], [82]). Ainsi, chaque insecte se développe dans une seule graine et, dans les régions d'origine, chaque espèce est spécialisée dans l'exploitation d'essences du même genre végétal. La reconnaissance de l'hôte est sous-tendue par des signaux visuels et olfactifs émis par les cônes de cèdre ([71], [83]). Par ailleurs, les *Megastigmus* disposent d'une batterie de traits adaptatifs permettant à leurs populations de faire face aussi bien à l'absence de graines sur une ou plusieurs années (diapause prolongée larvaire [90]), que de s'affranchir de la rencontre des sexes (parthénogénèse [81]). L'ensemble de ces mécanismes d'échappement constitue un avantage évident dans les zones d'invasion, et contribue à l'intérêt de la construction d'un modèle.

Nous savons de plus que le cèdre est également une espèce végétale introduite, qui n'a été encore que peu colonisée par la faune autochtone (un seul autre prédateur, *M. pinsapinis*, qui semble avoir des capacités invasives inférieures à celles de *M. schimitscheki* [40]). Cela contribue à la simplicité du modèle étudié, tout comme le fait que *M. schimitscheki*, étant une espèce introduite, n'a pas de prédateur.

Nous utilisons ici deux modèles imbriqués. L'un sert à simuler la ponte et la phase larvaire, en particulier la diapause prolongée, l'autre permet de calculer la dispersion des adultes. C'est un modèle de réaction-diffusion, avec un terme de transport. Les modèles de réaction-diffusion sont souvent utilisés en dynamique des populations, en particulier dans les cas où l'on ne peut suivre les individus et observer leur mouvement (voir les livres de P. Turchin, [89] ou N. Shigesada et K. Kawasaki [85]). Rappelons que la dispersion passive se distingue du mouvement actif par la source d'énergie utilisée pour accomplir les déplacements. Ici, on parlera de dispersion "semi-passive", car l'insecte peut interférer avec la dispersion passive, en modifiant son altitude de vol, s'il se trouve près d'un bassin d'attraction (une forêt). Le mouvement se fait dans un espace à trois dimensions, (on se place sur une carte en deux dimensions, et on peut faire intervenir l'altitude sur des paramètres tels que la date d'émergence).

Nos objectifs initiaux étaient (i) de construire un modèle reproduisant fidèlement le comportement de *M. schimitscheki*, (ii) d'utiliser ce modèle pour approfondir la compréhension des processus d'invasion, et (iii) de construire un modèle pouvant éventuellement s'adapter à plusieurs espèces (le projet concernant en effet également l'aspect compétition cf. [40]), pour définir des critères plus ou moins favorables à l'expansion.

2 Aspect pratique, site et collecte des données

Des collectes annuelles standardisées de cônes mûrs ont été organisées par les agents de l'INRA Avignon depuis 1994 dans la plupart des cédraies du Sud-Est de la France, le nombre de cônes collectés variant de 8 à 139 selon l'importance de la fructification. Cette dernière a été estimée par les services de l'Office National des Forêts.

Après récolte, les cônes ont été désarticulés et leurs graines radiographiées afin d'évaluer les proportions respectives de graines pleines, attaquées et vides, qui se séparent aisément par radiographie. A part *M. schimitscheki*, une autre espèce introduite, *M. pinsapinis*, peut infester les graines et apporter une confusion à l'examen des radiographies. Les graines attaquées seront donc surveillées en éclosoir jusqu'à l'émergence des adultes, afin d'apprécier la proportion respective de chaque espèce, et d'estimer les dégâts imputables à *M. schimitscheki*.

Le calcul du pourcentage d'attaque se fait de la façon suivante :

$$\%\text{attaque} = \frac{\text{nombre de graines attaquées}}{\text{nombre de graines attaquées} + \text{pleines}} \times 100, \tag{1}$$

on fait donc implicitement l'hypothèse, discutable (voir la section 6.3), que les insectes n'attaquent que les graines pleines (pollinisées).

Les données concernant les vents sont fournies par des stations Météo France réparties sur le territoire étudié.

3 Formulation d'un modèle spécifique

3.1 Hypothèses comportementales utilisées dans la construction du modèle

3.1.1 Hypothèses simplificatrices

Nous avons mis en lumière les aspects importants du problème, et ceux qui sont d'une importance moindre, pour établir quels sont les paramètres qui doivent être pris en compte dans la construction du modèle, et ceux qui peuvent être écartés. Cela nous permet finalement d'avoir une formulation plus claire du problème (étape décrite par Turchin [89]).

Ainsi, les données dont nous disposons [40] rendent compte d'un sex-ratio à peu près constant au moment de l'éclosion (environ 50% de mâles, 50% de femelles), et le mécanisme de parthénogénèse ne rend pas indispensable la fécondation des femelles [81]. Nous ne modélisons donc pas les phénomènes de fécondation et les interactions mâle/femelle. Par ailleurs, si les variations des dates d'émergence en fonction de l'altitude pourraient facilement être intégrées, nous ne les prenons pas en compte dans les tests numériques, afin de réduire le nombre de données à entrer, et de clarifier le modèle.

Nous représentons ci-dessous un cycle de vie simplifié de l'insecte, sur deux ans, tel que nous l'avons pris en compte pour l'élaboration du modèle.

3.1.2 Comportement

Les insectes éclosent progressivement sur une période de 12-15 jours [40]. Le temps de vol estimé d'un insecte est d'environ 10 jours [59] ; en outre, les *Megastigmus* ne volent que durant la journée [80] ce qui ramène à environ 7 jours de temps de vol effectif. Nous supposons qu'ils

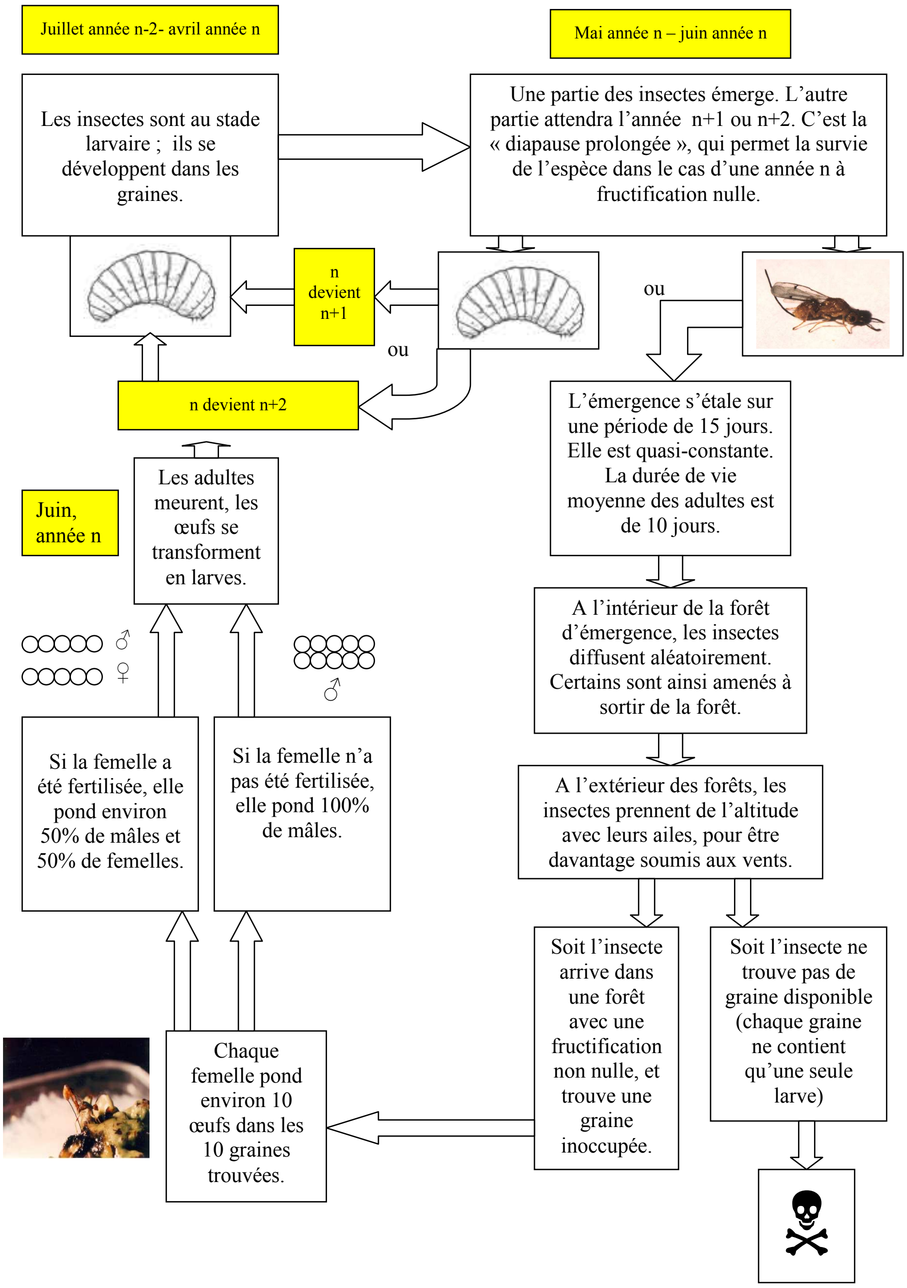


FIG. 1 – Cycle de vie simplifié

diffusent au sein de la forêt puis, quand les mécanismes de diffusion les amènent à en sortir, ils se servent de leurs ailes pour s'élever et être ainsi soumis à des vents plus forts. Il est en effet évident que le transport de l'insecte par le vent n'est pas passif (même si ses ailes ne lui permettent pas d'aller à son encontre). En effet, on note expérimentalement (A. Chalon, données non publiées) qu'un grand nombre d'insectes est retrouvé à de grandes distances (100 km) du point source d'année en année, ce qui n'est pas compatible avec une "dispersion passive sur une longue distance" par le vent, en comparaison avec les distances parcourues par des graines (voir par exemple l'article de Nathan *et al.* [74]). Par ailleurs, cette hypothèse "d'élévation" des insectes, et les distances parcourues relevées expérimentalement, concordent avec les résultats décrits par Tackenberg *et al.* dans [88] pour les graines. Il est en effet souligné dans cet article que la dispersion sur une longue distance est en grande partie conditionnée par les vents ascendants. Arrivé à une certaine distance d_a d'une forêt l'insecte réduit alors son altitude afin d'être moins soumis au vent (attraction visuelle et olfactive [71], [83]) et de s'approcher de la foret par diffusion. Au cours de la période de vol, nous supposerons que le coefficient de mortalité, X est constant.

Le cycle de développement simplifié est résumé à la figure 1.

3.2 Modélisation de l'évolution pendant la période de vol

3.2.1 Equation de réaction-diffusion

Afin de modéliser le déplacement décrit dans les sections précédentes, nous utilisons l'équation de réaction-diffusion suivante

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) - v_* V(t,x)\cdot\nabla u(t,x) + f(t,x) - X(t,x)u, \tag{2}$$

pour $t \in [0, N_j]$ (où N_j est le nombre de jours total où des adultes sont présents), et $x \in \Omega$, où Ω est un ouvert suffisamment grand de $\mathbb{R}^2$, et

$$u \equiv 0 \text{ sur } \partial\Omega.$$

Explicitons chaque terme de l'équation (2), et donnons sa valeur.

Le terme $f(t,x)$ correspond à un terme de "source", modélisant l'émergence progressive des insectes. Les résultats de [40] (Fig. 2), montrent un taux d'émergence quotidien des femelles (rappelons que nous simulons le déplacement des femelles) à peu près constant sur la période N_{em}. Soit $u_0(x)$ le nombre d'insectes prêts à éclore, avant l'émergence, à la position $x \in \mathbb{R}^2$. Alors,

$$f(t,x) = \frac{u_0(x)}{N_{em}} \text{ pour } t \in [0, N_{em}], \text{ et } f(t,x) = 0 \text{ pour } t > N_{em}. \tag{3}$$

Nous devons ensuite ajouter un terme de mortalité quotidienne $X(t,x)$ durant la période de vol. Ce taux étant supposé constant, nous avons

$$X(t,x) = \frac{1}{\text{durée moyenne de vol}} \text{ pour } x \text{ à l'extérieur d'une forêt}, \tag{4}$$

nous supposons de plus que

$$X(t,x) = 0 \text{ pour } x \text{ dans une forêt}, \tag{5}$$

la ponte n'étant en effet prise en compte qu'à la fin de la période de vol, nous devons faire comme si les insectes arrivés dans une forêt y restaient jusqu'à la fin de la période de vol.

Nous avons donc précisé le terme de réaction. Venons-en à la diffusion. La valeur de D étant ici difficile à estimer pour des insectes qui sont trop petits pour être marqués et recapturés, il ne nous paraît pas judicieux de faire varier D avec x et t (au moins dans un premier temps). De plus, des modèles pour d'autres types d'insectes se sont montrés efficaces avec des valeurs de transport variables et un terme de diffusion fixe [9].

Le terme de transport est variable en temps et en espace, la variabilité en temps correspondant aux variations quotidiennes des vents. Concernant la variation en espace, nous prendrons en compte, en sus des vents mesurés, le critère d'éloignement d'une forêt. Nous supposerons en effet que l'insecte ne dispose pas de critères "décisionnels" lui commandant de sortir de la forêt quand celle-ci est saturée en insectes, et qu'il n'en sort que par un mécanisme de diffusion aléatoire, et qu'ensuite, comme précisé dans la section 3.1.2, il s'élève pour être davantage soumis au vent, en fonction de son éloignement de la forêt. Nous supposons de plus qu'il prend de l'altitude de façon proportionnelle à sa distance de la forêt, jusqu'à une distance maximale, d_a. Pour préciser la force du vent à laquelle il est soumis, nous utilisons un résultat décrit par Nathan *et al.* [74], et utilisé pour calculer la force du vent à laquelle est soumise une graine pour une végétation rase. En effet, l'insecte s'élevant suffisamment haut, nous pouvons supposer qu'à l'extérieur de la forêt de cèdre, le milieu correspond plus à une végétation "rase" qu'à une forêt dense. Il est ainsi établi dans [74] que le profil du vent, qui décrit sa moyenne horizontale à chaque altitude, est logarithmique. La formule utilisée pour calculer V est donc :

$$V(t,x) = V_e(t,x)\min\{ln[d(x,foret)(e-1)/d_a+1],1\}, \tag{6}$$

où V_e est le vent enregistré, et $d(x, foret)$ la distance à la forêt de cèdre la plus proche. S'impose alors l'introduction d'un autre coefficient v_*. En effet, si la distance parcourue par les insectes est très supérieure à la "dispersion longue distance" passive de graines [74], elle est cependant inférieure à la distance totale parcourue par l'air en mouvement (le vent). Cela pourrait s'expliquer de différentes façons : pauses faites par l'insecte durant sa période de vol ou encore turbulences verticales du vent réduisant l'altitude de vol de l'insecte. Cependant, ces hypothèses ne sont pas vérifiables, et nous serons amenés à estimer le coefficient v_* dans la section 4.

3.2.2 Résolution numérique

La méthode utilisée est du type "directions alternées" : c'est la méthode des pas fractionnaires, avec un schéma d'Euler implicite et un décentrage en amont pour le terme d'ordre 1. Cette méthode est proposée par Godunov dans [7] et Yanenko dans [94] pour l'équation de la chaleur. Cela s'écrit

$$\begin{aligned}
\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{dt} &= D\frac{u_{i+1,j}^{n+1/2}-2u_{i,j}^{n+1/2}+u_{i-1,j}^{n+1/2}}{h_1^2} \\
&+v_*\left(V_{i,j}^h-|V_{i,j}^h|\right)\frac{u_{i+1,j}^{n+1/2}-u_{i,j}^{n+1/2}}{2h_1} \\
&-v_*\left(V_{i,j}^h+|V_{i,j}^h|\right)\frac{u_{i,j}^{n+1/2}-u_{i-1,j}^{n+1/2}}{2h_1}+f_{i,j}^n-X_{i,j}u_{i,j}^{n+1/2}, \text{ et} \\
\frac{u_{i,j}^{n+1}-u_{i,j}^{n+1/2}}{dt} &= D\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h_2^2}+v_*\left(V_{i,j}^v-|V_{i,j}^v|\right)\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}}{2h_2} \\
&-v_*\left(V_{i,j}^v+|V_{i,j}^v|\right)\frac{u_{i,j}^{n+1}-u_{i,j-1}^{n+1}}{2h_2}+f_{i,j}^n-X_{i,j}u_{i,j}^{n+1},
\end{aligned} \tag{7}$$

où V^h et V^v sont respectivement les composantes horizontales et verticales du vent défini par (6), dt, h_1, h_2 les pas de temps et d'espace.

Il est prouvé qu'un tel schéma avec seulement des termes d'ordre 2 est inconditionnellement stable au sens de Fourier [84], un calcul de la stabilité avec des termes d'ordre 1 étant plus fastidieux. De plus, si la stabilité n'est pas prouvée théoriquement, les différents tests numériques effectués au cours de ce travail, et dans [84], ont suggéré que la méthode était stable.

3.3 Modélisation de la phase ovo-larvaire

Soit $u(x)$ le nombre d'insectes à la fin de la période de vol, l'année n. Chaque insecte arrivé dans une forêt de cèdre pond un nombre ϵ d'oeufs dans la limite des ressources disponibles. Ainsi,

$$o(x) = \min\{g(x), \varepsilon u(x)\},$$

où $o(x)$ est le nombre d'oeufs pondus et $g(x)$ le nombre de graines disponibles. Soit d_1 le taux de diapause prolongée à 1 an et d_2 le taux de diapause prolongée à 2 ans. Alors le nombre de nouveaux adultes en mai de l'année $n+2$ sera

$$u(0, x) = o(x)(1 - d_1 - d_2) + o_{-1}d_1 + o_{-2}d_2,$$

où o_{-1} et o_{-2} sont le nombre d'oeufs pondus les 2 années précédentes.

La mortalité durant cette phase est donc supposée nulle, comme cela a été montré pour des espèces voisines [59]. Il n'existe en effet actuellement ni parasite ni prédateur dans la zone d'invasion.

4 Estimation des paramètres

Certains paramètres vont pouvoir être estimés directement, d'autres vont nécessiter un recours au calcul numérique.

Calculons d'abord la valeur du coefficient de diffusion. Dans [59], pour une espèce proche, on a $D = 1km^2/jour$. Dans le livre de N. Shigesada et K. Kawasaki [85], pour d'autres espèces d'insectes (piéride du chou, qui est cependant un meilleur voilier), des valeurs différentes sont citées : $D = 0.08 - 0.46km^2/jour$. Afin de calculer la valeur de D la plus en adéquation avec nos résultats expérimentaux, nous allons utiliser une formule de [85] (p. 38), valable dans le cas où D est le seul paramètre influant sur la dispersion des insectes. Rappelons que dans notre modèle les vents n'ont pas d'incidence sur le mouvement des insectes à l'intérieur des forêts. La formule citée ci-dessus est

$$D = \frac{< r >^2}{\pi t}, \tag{8}$$

où $< r >$ est le rayon moyen parcouru par les insectes, à partir du point source, pendant le temps t. En utilisant les mesures effectuées dans la forêt du Lubéron, entre Oppede et Ménerbes, nous estimons les déplacements moyens des insectes à $< r >= 1km/jour$. Nous retrouvons des valeurs similaires sur les autres sites de mesure. Ainsi, la formule (8) donne

$$D = 0.32km^2/jour,$$

valeur proche de celle mentionnée dans [85] pour la piéride du chou (d'après des travaux de Andow *et al.* [4]).

La valeur du taux de mortalité durant la période de vol, $X(t, x)$, peut être calculée directement avec les formules (4) et (5), en utilisant la durée de vol moyenne de 10 jours [59]. Le paramètre N_{em} est également connu, et sa valeur est comprise entre 6 et 11 jours [40]. Nous prendrons en fait pour les calculs 2/3 de la durée de vol moyenne et 2/3 du nombre de jours d'émergence pour tenir compte de l'activité uniquement diurne de l'insecte [80]. Le taux de fécondité ε est voisin de 10 oeufs par femelle (voir [59] et résultats récents de A. Chalon, données non publiées). Les taux de diapause prolongée que nous utiliserons sont de 5% à un an et 15% à deux ans. Ces valeurs nous ont été transmises par les biologistes. Il semble cependant que ces taux varient d'une année sur l'autre, en fonction de la fructification, et puissent prendre des valeurs plus importantes (voir la section 6.1.2). Des expériences sont en cours pour tenter de mieux comprendre comment varient ces paramètres.

Au sujet des vents, rappelons que la direction ainsi que la vitesse du vent (donc les paramètres V_h et V_v) proviennent de mesures directes de stations Météo France relevées chaque jour (à 10 mètres du sol) durant la période de vol. Nous devons donc uniquement estimer le coefficient v_* et le paramètre d_a. La valeur du paramètre d_a, correspondant à la distance à partir de laquelle les forêts n'ont plus d'influence sur l'altitude de l'insecte, sera fixée ici à $10km$ (après discussion avec les biologistes).

Pour estimer v_*, nous allons utiliser une méthode générale décrite par P. Turchin dans son livre [89], d'après les travaux de Banks et al. [8]. Nous devons commencer par faire une hypothèse initiale sur la valeur de v_*. Nous calculons pour cela le rapport entre la distance maximale mesurée d'avancée du front d'une année sur l'autre et la distance maximale parcourue par les vents dans une même direction pendant une période de 15 jours (durée de vol effective de 6.5% des insectes). Nous obtenons ainsi $v_* = 0.012$ (les données utilisées pour ce calcul sont relatives aux sites d'Aurel et de Sisteron).

Après implémentation du modèle (avec Matlab®), et résolution numérique, nous confrontons les prédictions aux données expérimentales pour la distance maximale parcourue par le front d'une année sur l'autre, afin d'ajuster la valeur du coefficient v_*.

A partir des données dont nous disposons, et en les confrontant aux prédictions du modèle, nous pouvons donner l'intervalle suivant : $v_* = 0.04 - 0.08$. Ne disposant pas de la valeur de la fructification sur les nouvelles zones d'invasion, utilisées pour ce calcul, nous ne pouvons donner de valeur plus précise.

5 Tests numériques

Nous nous penchons sur l'influence de différentes caractéristiques biologiques d'une espèce sur sa propagation, sous plusieurs conditions de vents et de répartition des forêts de cèdre. Ainsi, sont étudiés i) l'influence de la durée de la période d'émergence, ii) l'influence du taux de diapause prolongée et iii) l'effet de la présence de cèdres isolés entre les forêts. Cela nous permet de souligner le rôle de ces facteurs, et de voir dans quelles situations ils sont avantageux.

Pour chacun de ces points, nous effectuons plusieurs tests numériques, qui nous permettront de répondre aux questions ci-dessus, et de vérifier dans la section 6.1.1 que les résultats fournis par le modèle ont une interprétation "logique" du point de vue du biologiste.

Pour les différents tests numériques effectués dans cette section, nous n'utilisons pas les coordonnées réelles des plantations de cèdre, mais des forêts "modèles", permettant une interprétation plus facile des tests. De même, les vents seront choisis indépendamment des vents réels. Par ailleurs, nous faisons varier certaines caractéristiques des insectes, telles que le taux de fécondité ou la diffusion, afin de souligner l'influence des autres paramètres sur la propagation.

5.1 Test 1 : Influence de la durée de la période d'émergence

Nous étudions ici l'influence de la longueur de la période d'émergence sur l'étendue de la propagation des insectes, suivant notre modèle.

5.1.1 Vents constants

Dans un premier temps, les vents sont supposés constants durant l'émergence et d'année en année, et les forêts sont disposées comme sur la figure 2.

La fructification est uniforme (10hl de graines par plantation), et le pourcentage d'attaque initial est nul partout sauf en $d7$ (flèche sur la figue 2), où il est de 50%. La figure 3 présente la dispersion suivant la longueur de la période d'émergence, sur une durée totale de 6 ans.

Conclusion : la longueur de la période d'émergence n'a pas d'influence significative sur la dispersion dans le cas de vents constants.

5.1.2 Direction des vents variable

Ici, les vents sont choisis constants orientés vers le Nord pendant 7 jours, puis vers le Sud (avec la même vitesse moyenne). Les forêts et les vents sont représentés sur la figure 4.

A nouveau, le pourcentage d'attaque initial est nul partout sauf en $d13$ (flèche sur la figure 4). Le taux de fécondité $\varepsilon = 30$ pour souligner les écarts. Les simulations (toujours sur une période de 6 ans) sont présentées sur la figure 5.

Conclusion : dans le cas d'une émergence sur 7 jours, progression vers le nord légèrement plus importante ; cependant, dans le cas d'une émergence sur 14 jours, 7 parcelles sont infestées contre 4 dans le premier cas.

5.1.3 Force des vents variable

Cette fois, les vents sont orientés vers le Nord, mais soufflent la deuxième semaine à vitesse moyenne double de la première. En effet, le coefficient pour le terme de transport est $v_* \|V\| = 3km/jour$ la première semaine et $v_* \|V\| = 6km/jour$ la deuxième semaine (voir la figure 6 pour le domaine utilisé).

Le pourcentage d'attaque initial est nul partout sauf en $d2$ (flèche sur la figure 6). A nouveau, le taux de fécondité est de $\varepsilon = 30$. Les simulations sont données sur la figure 7.

Conclusion : Alors que les insectes dont la période d'émergence ne s'étale que sur 7 jours ne parviennent pas à atteindre les parcelles au Nord, ceux qui ont une période d'émergence de 14 jours colonisent deux parcelles supplémentaires au Nord.

5.2 Test 2 : Influence du taux de diapause prolongée

Dans ce test, nous nous intéressons à la propagation sur 6 ans d'une espèce, en fonction de l'existence d'une période de diapause prolongée.

5.2.1 Cas de la fructification constante

La fructification est ici constante en temps (d'année en année) et en espace. C'est une hypothèse hautement improbable en milieu naturel, mais elle va nous permettre de voir si la diapause prolongée fait varier la dispersion spatio-temporelle indépendamment de la fructification. Nous sommes dans les conditions décrites par la figure 2. Le taux de fécondité sera supposé égal à 30, la diffusion $D = 0.25km^2/jour$ et le coefficient de transport $v_*\|V\| = 2km/jour$.

Les tests numériques sont représentés sur la figure 8.

Conclusion : Dans le cas sans diapause prolongée, la zone envahie est légèrement plus importante.

5.2.2 Cas de la fructification variable

Dans ce test, certaines parcelles pourront avoir une fructification nulle suivant l'année. Nous nous plaçons toujours dans la configuration décrite par la figure 2. Pour la fructification, l'hypothèse faite est décrite par la figure 9. Ainsi, la fructification est supposée uniforme ($10hl$ par parcelle) la première année puis nulle sur certaines parcelles les années suivantes. Nous verrons dans la section 5.2.4 que cette hypothèse est plausible d'un point de vue biologique. Les résultats sont présentés sur la figure 10.

Conclusion : Cette fois, la propagation est plus étendue notamment au Nord dans le cas de l'espèce avec diapause prolongée.

5.2.3 Accidents avec fructification nulle partout pendant 1 an ou 2 ans

Le devenir de l'espèce qui ne fait pas de diapause prolongée est évident. Le cycle du *M. schimitscheki* étant de 2 ans, si la fructification est nulle l'année n, il n'y aura pas d'insecte pour toutes les années $n + 2k$, $k \in \mathbb{N}$. Si elle est nulle l'année n et l'année $n + 1$, l'espèce disparaît. Le modèle retranscrit fidèlement ces résultats. Intéressons-nous maintenant à l'influence de l'existence d'une telle année à fructification nulle dans le cas d'une espèce avec diapause prolongée. Le taux de fécondité vaut $\varepsilon = 30$, la diffusion $D = 0.25km^2/jour$ et le terme de transport $v_*\|V\| = 2km/jour$. Le taux de diapause prolongée est supposé égal à 5% à un an et 15% à 2 ans. Nous sommes encore dans la situation décrite par la figure 2, avec une fructification uniforme de $10hl$ par parcelle. La propagation est étudiée sur une durée de 6 ans. Le test numérique effectué est représenté sur la figure 11. On y a supposé que le cycle de vie de l'insecte était de 1 an (au lieu de 2 dans le cas de *M. schimitscheki*), afin d'obtenir des différences plus significatives (cela revient à supposer que la fructification est nulle pendant 2 ans, sur une période totale de 12 ans)

Conclusion : la présence d'une année où la fructification est nulle n'a que peu d'incidence sur la propagation d'une espèce avec diapause prolongée.

5.2.4 Cas concrets

Nous présentons ici quelques données mesurées dans les cédraies au Mont-Ventoux afin d'illustrer l'intérêt des tests sur la diapause prolongée et les arbres isolés. D'après les résultats de la figure 12, la fructification peut en effet être nulle certaines années sur une zone étendue (forêt du Ventoux : $800ha$), ou encore nulle à partir d'une année sur zone ici moins étendue (forêt d'Oppede : $20ha$).

5.3 Test 3 : Influence des arbres isolés

Deux configurations différentes de parcelles sont étudiées (cf. figure 13). Ainsi, nous nous penchons sur la question de l'influence de la présence d'arbres isolés entre deux zones forestières. Les vents sont donnés par la figure 2 ; le taux d'attaque initial est supposé nul partout sauf en $d2$ où il est de 50%.

5.3.1 Cas où le transport est grand devant la diffusion (régions venteuses)

Dans ce cas, le terme de transport a une norme $v_*\|V\| = 5km/jour$, et la diffusion vaut $D = 0.25km^2/jour$. De plus, $\varepsilon = 15$. Les résultats obtenus sont présentés sur la figure 14.

Conclusion : la propagation est plus rapide dans le cas sans arbre isolé.

5.3.2 Cas où le rapport diffusion/transport est plus grand

Cette fois, le terme de transport a une norme $v_*\|V\| = 2km/jour$, et la diffusion vaut $D = 3km^2/jour$. Ici, $\varepsilon = 50$. Les résultats obtenus sont présentés sur la figure 15.

Conclusion : ici, la propagation est plus importante et les insectes arrivent à atteindre les parcelles situées au Nord, dans le cas avec arbres isolés ; ils n'y parviennent pas dans l'autre cas.

6 Discussion

6.1 Validation du modèle

6.1.1 Interprétation des résultats donnés par les tests.

Avant d'accorder de l'importance aux résultats découlant de la section 5, nous pouvons nous demander si le comportement décrit par le modèle est raisonnable ou non. Nous allons donc vérifier que chacun des résultats numériques obtenus dans la section 5 a une interprétation biologique "logique".

Intéressons-nous d'abord aux résultats numériques obtenus concernant la période d'émergence. Pour des vents constants (section 5.1.1), on comprend aisément que seul le nombre total d'insectes émergents joue un rôle sur la dispersion. De même, pour les vents variables en direction (section 5.1.2), dans le cas d'une période de 7 jours, l'envol d'un nombre plus important d'insectes permettrait d'obtenir une colonisation plus significative au Nord ; ensuite, quand la deuxième semaine les vents seraient dirigés vers le Sud, les insectes encore en vie seraient redéplacés vers le Sud, mais ne vivraient pas assez longtemps pour coloniser des parcelles au Sud. Dans le cas d'une émergence sur 14 jours, le vent soufflant vers le Sud amènerait les insectes ayant récemment émergé vers les parcelles situées au Sud. De plus certains insectes, d'abord déportés vers le Nord, puis vers le Sud seraient amenés par diffusion vers les parcelles à l'Est et à l'Ouest de la zone initiale d'envol. Finalement, nous pouvons également interpréter de la façon suivante les résultats obtenus pour un vent variable en intensité (section 5.1.3) : les insectes qui émergeraient la deuxième semaine bénéficieraient de vents plus forts qui leur permettraient de franchir la zone sans cèdre située entre les parcelles $d2$ et $d22$.

Ensuite, concernant la diapause prolongée, dans le cas d'une fructification constante (section 5.2.1), les résultats obtenus peuvent s'interpréter de la façon suivante : au niveau de la zone de front, chaque année, la diapause prolongée ferait “perdre” 50% des adultes émergents (avec les données utilisées dans ce test). On comprend donc pourquoi la zone d'invasion serait légèrement en recul dans le cas de la population avec diapause prolongée. Maintenant, dans le cas d'une fructification variable (section 5.2.2), les résultats peuvent s'interpréter comme suit : la première année, les individus se propageraient sur les premières parcelles situées au Nord ($d11, d12, d13$). L'année suivante, dans le cas sans diapause prolongée, 100% des insectes émergeraient et, comme la fructification serait nulle sur une bande au Nord ($d11, d12, d13$ et $d16, d17, d18$), les insectes mourraient sans y avoir pondu. L'année suivante, les insectes qui seraient entrés en diapause prolongée les deux années précédentes émergeraient comme adultes (sur les parcelles $d11, d12, d13$), et le vent leur permettrait d'aller envahir les parcelles situées plus au Nord ($d16, d17, d18$), alors que dans le cas où la diapause prolongée serait nulle le vent trop faible rendrait la zone à fructification nulle ($d11, d12, d13$) infranchissable. On peut noter une progression légèrement plus importante au Sud dans le cas avec diapause prolongée, due au phénomène décrit dans le cas de la fructification constante. Pour le cas où la fructification présenterait des “accidents” (section 5.2.3) on comprend que l'existence d'une année avec fructification nulle, reviendrait, ici, à diviser le nombre d'adultes total par 5, ce qui n'aurait que peu d'influence sur la position du front. De plus, cela ferait “perdre” un an en terme de propagation.

Finalement, au sujet des arbres isolés, quand le terme de transport est important devant la diffusion (section 5.3.1), les résultats obtenus s'interprètent comme suit : dans le cas avec arbres isolés, les insectes réduiraient leur altitude pour s'approcher des arbres, et seraient de ce fait soumis à des vents moins forts, ce qui pourrait expliquer une propagation moins rapide. Enfin, avec un rapport diffusion/transport plus grand (section 5.3.2), la zone sans arbre serait trop grande pour être franchie en une seule année au vu de la force des vents. Les arbres isolés serviraient donc de relais pour l'invasion des zones plus au Nord. Cependant, pour obtenir une différence, nous avons dû choisir un taux de natalité $\varepsilon = 50$, car le faible nombre d'insectes émergents sur les arbres isolés mettrait du temps avant de former une population significative (donc mesurable) envahissant les parcelles situées au Nord.

Tous les résultats numériques obtenus dans la section 5 peuvent donc être reliés à une interprétation logique du point de vue biologique. De plus, ces résultats concordent avec les prévisions des biologistes (avant la construction du modèle). Cela nous permet de penser que le comportement qualitatif donné par le modèle est raisonnable. Pour tester le modèle de façon quantitative, nous devons comparer les résultats mesurés sur la zone étudiée avec les prévisions du modèle. C'est le but de la section suivante.

6.1.2 Comparaison des prévisions avec les mesures effectuées

Les données dont nous disposons nous permettent uniquement de tester le modèle pour la période 2001-2003.

Nous connaissons le pourcentage de graines desquelles des adultes de *M. schimitscheki* émergent en mai 2001 (i) à partir de la récolte 2000 (sans diapause prolongée), et (ii) à partir de la récolte 1999 (un an de diapause prolongée). Nous savons que la période d'émergence a débuté entre les 5 et 7 mai 2001, et s'est étalée sur une période d'environ 15 jours (mesures effectuées en laboratoire, pouvant varier suivant l'altitude). La connaissance de la fructification et des pourcentages d'attaques nous permet d'estimer le nombre d'insectes émergents sur cette période. Ces paramètres, ainsi que la donnée des vents, vont permettre de calculer la position et le nombre théorique d'insectes correspondant à l'attaque de la récolte 2002. Nous allons comparer ces résultats avec ceux mesurés (obtenus lors de l'émergence en laboratoire : en mai 2003).

Position du front

Nous ne connaissons pas la fructification dans les zones nouvellement envahies. Nous ne pouvons donc pas faire de comparaison en termes de nombre d'insectes ou de pourcentage d'attaque sur ces zones. Nous comparons donc la position du front d'invasion donnée par le modèle avec celle observée expérimentalement. Les résultats sont présentés sur la figure 16.

Conclusion : les parcelles nouvellement envahies, d'après le modèle, sont Sisteron et Lure. Expérimentalement, seule la parcelle de Sisteron est nouvellement envahie en 2001. Plus précisément, des graines attaquées ont été récoltées en 2002 à Lure (correspondant, rappelons-le, à l'attaque de 2001), mais uniquement des *M. pinsapinis* en ont émergé en 2003. L'émergence en mai 2004 nous dira si des *M. schimitscheki* en diapause prolongée étaient présents dans ces graines. D'après les prévisions du modèle, les parcelles de Gap, Saou, Barjac, Moulin, Saumon ne sont pas envahies. Ces résultats sont en adéquation avec les faits. Du point de vue de la position du front, les résultats fournis par le modèle que nous proposons semblent donc cohérents.

Taux d'attaque

Rappelons que nous ne disposons de la fructification que sur les zones déjà colonisées par *M. schimitscheki* en 1999 et en 2001. La fructification en 1999, ainsi que les taux d'attaque mesurés en 2001, nous permettent d'estimer le nombre d'insectes prêts à s'envoler en mai 2001. Nous utilisons notre modèle pour prédire le nombre d'insectes présents sur chaque zone lors de la récolte 2003. La carte géographique telle que nous l'avons numérisée pour les calculs est représentée sur la figure 17. Les résultats du modèle (M), ainsi que le pourcentage de graines desquelles émergent des *M. schimitscheki* en 2003 (S) et le taux d'attaque mesuré à la radiographie (R) sont présentés sur la figure 18.

Le nombre de parcelles sur lesquelles il est possible de comparer le taux d'attaque prévu avec celui mesuré n'est pas suffisant pour permettre une réelle analyse statistique. Cependant, nous pouvons tirer de ces résultats quelques conclusions. Ainsi, la courbe représentant les taux d'attaque prévus par le modèle et celle représentant les taux d'émergence de *M. schimitscheki* ont une allure très semblable. Globalement, la courbe M est au dessus de S, avec une certaine conservation des proportions. Rappelons que M correspond au taux de graines attaquées, et inclut donc les insectes en diapause prolongée. Cela pourrait justifier la relation $M > S$. En effet, R est très supérieur à S, et si cette différence correspond en partie à la présence de l'autre ravageur des graines de cèdre, *M. pinsapinis*, on peut supposer qu'elle est également due à un nombre conséquent, et supérieur aux valeurs que l'on a utilisées, de *M. schimitscheki* en diapause prolongée. Ainsi, nous observons que, là où les taux d'attaque ne sont pas trop faibles, le rapport M/S est d'environ 3, ce qui correspondrait, en supposant que les résultats donnés par notre modèle sont vrais, à un taux total (sur deux ans) de diapause prolongée d'environ 60%. La courbe D présente les résultats que nous aurions obtenus avec un tel taux de diapause en 2001. Cette fois, on note une relation étroite avec les résultats mesurés.

6.2 Résultats sur le comportement de l'insecte

Dans cette section, nous résumons les résultats obtenus à l'aide tests numériques de la section 5, qui vont permettre de mieux comprendre le rôle joué par certaines caractéristiques de *M. schimitscheki*, et les avantages qu'elles peuvent éventuellement lui conférer par rapport à d'autres espèces de *Megastigmus*.

D'après la section 5.1, si la longueur de la période d'émergence n'a pas d'impact significatif sur la propagation des insectes indépendamment des vents, son influence est importante dans le cas de vents variables. Ainsi, dans une situation réelle, où les vents sont variables en direction et en vitesse, une période d'émergence plus longue permet une dispersion sur un territoire plus étendu. D'un point de vue de la compétition entre plusieurs espèces pour une même "niche", on peut ajouter, au sujet de la période d'émergence, que la date d'émergence joue également un rôle important, l'insecte émergeant le plus tôt ayant un avantage sur les autres.

On constate également, d'après les tests numériques de la section 5.2 que, comme pour la longueur de la période d'émergence, la diapause prolongée est un avantage en terme de propagation (et de survie) dans le cas d'un environnement présentant une certaine "variabilité". Par contre, dans le cas d'un environnement (très peu probable) où la fructification serait constante, elle fait prendre un léger retard à la position du front des insectes.

Les résultats de la section 5.3 nous montrent également que la présence d'arbres isolés n'est pas toujours un avantage. Cela dépendra de la région (venteuse ou non), et des caractéristiques de l'insecte (diffusion plus ou moins élevée).

Enfin, les tests numériques effectués pour valider le modèle (section 6.1.2) nous laissent penser que le taux de diapause prolongée peut être supérieur à celui qui nous avait été initialement transmis par les biologistes. Des expériences en cours devraient bientôt permettre d'affiner la valeur de ce taux de diapause.

6.3 Limites

Si les résultats des sections précédentes semblent encourageants, nous devons malgré tout souligner les limites du modèle.

D'abord en termes de prévisions. La modélisation du nombre d'insectes émergents l'année n nécessite de connaître (i) la fructification des années $n-2$ et précédentes et (ii) les données météorologiques du mois de mai de l'année $n-2$. Si l'on peut espérer estimer la fructification à l'aide d'un autre modèle, il est impossible de prévoir la direction et la force des vents d'une année sur l'autre (sauf peut-être pour certaines configurations de terrain). Ainsi, les prévisions d'invasion ne peuvent se faire que sur deux ans.

D'autres limites proviennent d'un manque de connaissances de certaines caractéristiques de l'insecte. Ainsi, nous n'avons pas encore d'estimation fiable des taux de diapause prolongée. De même, des expériences sont en cours pour apprécier si les insectes peuvent pondre dans les graines non pollinisées et s'y développer jusqu'à l'âge adulte, en bloquant le processus de dégénérescence observé dans les graines non pollinisées (comme c'est le cas pour d'autres espèces). Les taux d'attaque seraient donc plus faibles que ceux utilisés ici.

Enfin, nous devons noter que certains facteurs n'ont volontairement pas été pris en compte dans ce travail de modélisation. Par exemple, la competition interspécifique avec *M. pinsapinis*, qui peut jouer un rôle dans les forêts où le taux d'attaque est proche de la saturation. Cependant, les résultats récents de [40] tendent à montrer que *M. schimitscheki* est un meilleur compétiteur que *M. pinsapinis*, avec notamment une émergence plus précoce. L'influence de la compétition interspécifique sur la dispersion de *M. schimitscheki* s'en trouverait donc diminuée.

6.4 Résultats à venir

Outre les expériences citées dans la section 6.3, l'utilisation de marqueurs moléculaires hypervariables, comme les microsatellites, qui est en cours à l'INRA, devrait permettre de typer les populations de chaque site d'attaque, et d'apprécier les zones d'origine des individus considérés. Ceci devrait apporter des éléments nouveaux pour la compréhension des processus de dispersion.

6.5 Adaptation à d'autres espèces vivantes

Le modèle que nous avons développé, ainsi que le logiciel associé (voir sur la figure 19 un aperçu de l'interface graphique) peuvent aisément s'adapter à d'autres espèces, notamment aux autres espèces invasives, s'attaquant à des végétaux exotiques introduits, comme la mineuse du marronnier d'Inde, *Cameraria oridhella*, qui tend à envahir l'ensemble de l'Europe occidentale à l'heure actuelle.

En outre, les limites citées dans la section 6.3 pourraient dans certains cas être atténuées. Par exemple, pour des insectes rampants ou des rongeurs, le terme de transport pourrait être moins variable (ex. : une pente ou une zone attractive). Le modèle pourrait également s'adapter au milieu marin, où le terme de transport (les courants marins) serait également plus facilement prévisibles que dans le cas que nous avons étudié (ex. : la Caulerpe). Ce type de modèle, avec quelques modifications quant à la prise en compte des variables, pourrait également s'appliquer à l'analyse et à la prévision des phénomènes d'expansion de certains insectes en relation avec le réchauffement climatique. Nous prévoyons ainsi de tester notre modèle dans le cadre d'un autre projet INRA en cours, s'intéressant à la progression en latitude et en altitude de la processionnaire du pin en France.

7 Figures

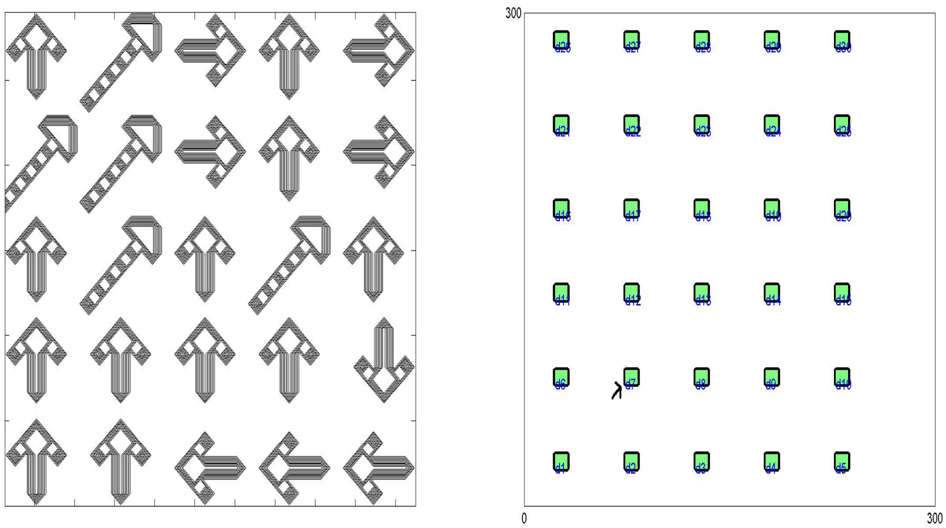


FIG. 2 – vents utilisés pour les tests 1.1, 2.1, 2.2, 2.3, 3.1, 3.2 et domaine utilisé pour les tests 2.1, 2.2, 2.3

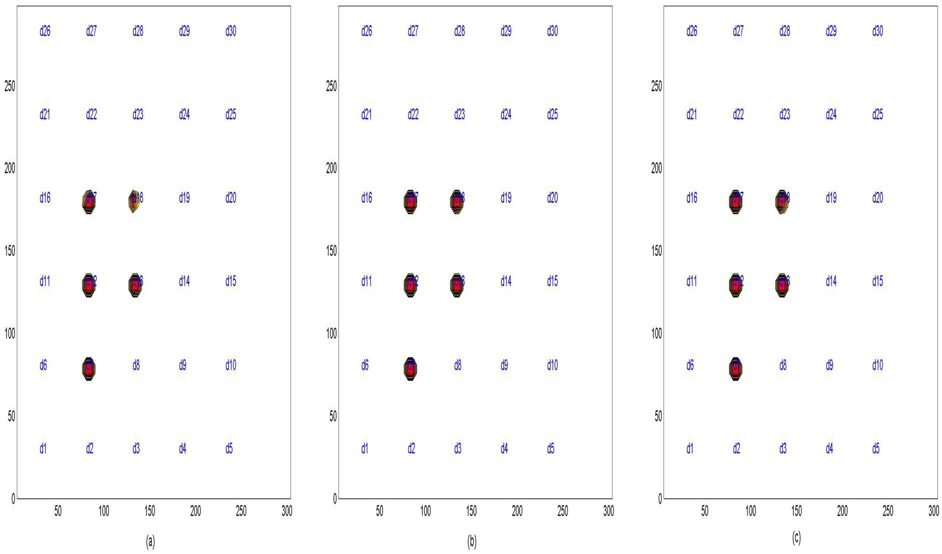


FIG. 3 – (a) : émergence sur 2 jours, (b) : émergence sur 5 jours, (c) : émergence sur 10 jours (test 1.1)

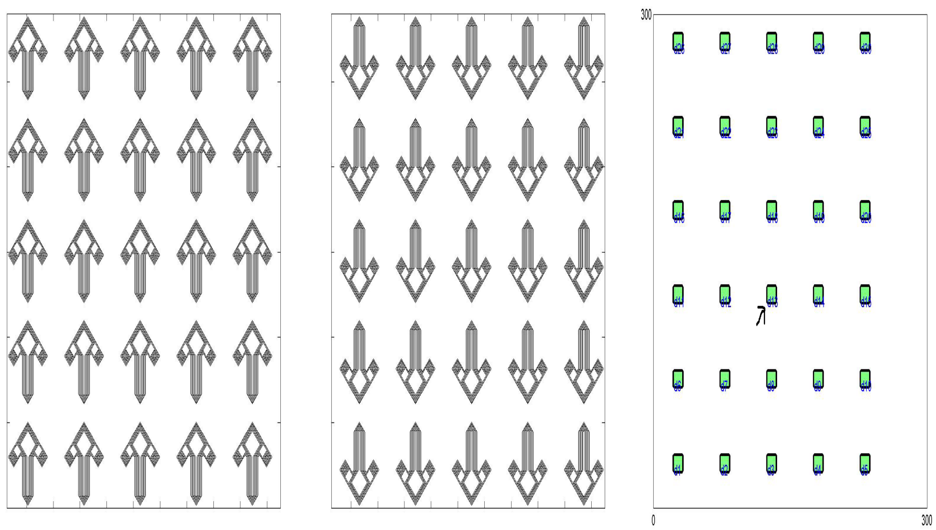


FIG. 4 – Vents pendant la première et deuxième semaine et domaine utilisé (test 1.2)

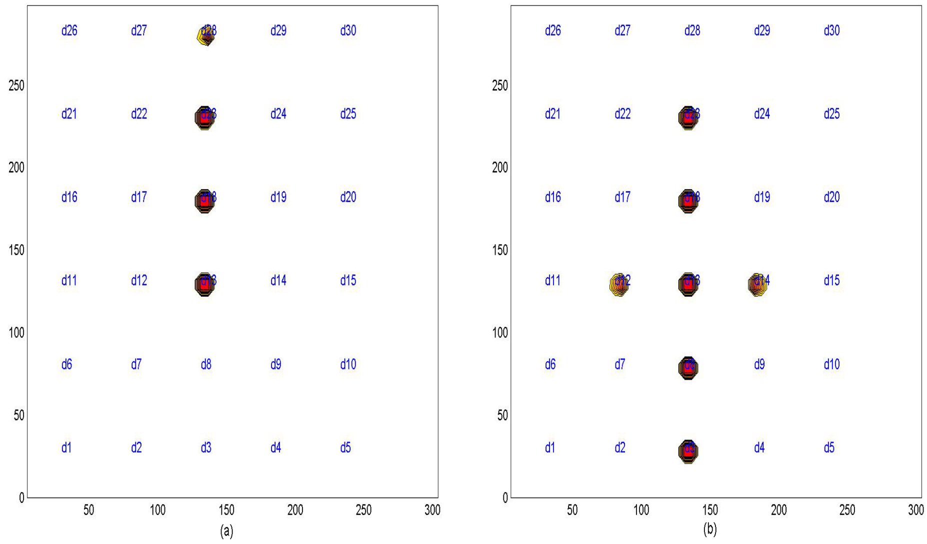


FIG. 5 – (a) : émergence sur 7 jours, (b) : émergence sur 14 jours (test 1.2)

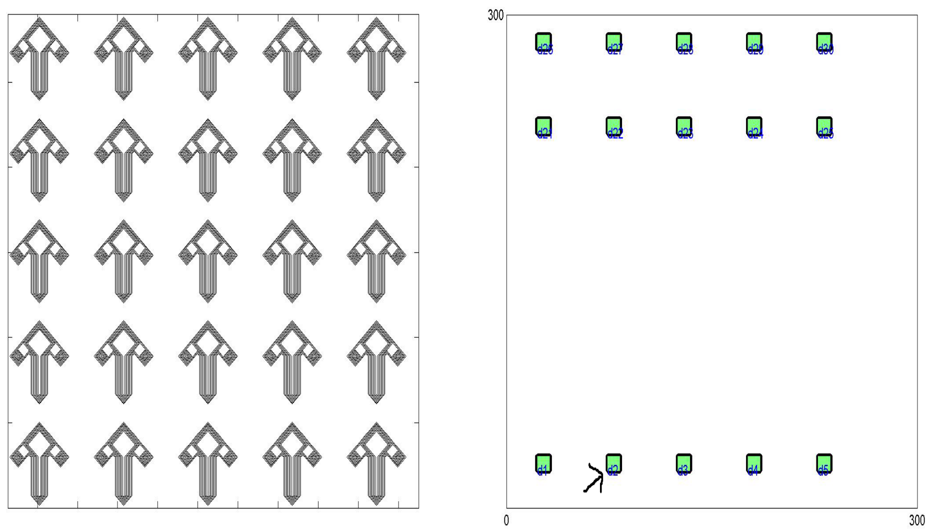


FIG. 6 – Direction des vents et domaine utilisé (test 1.3)

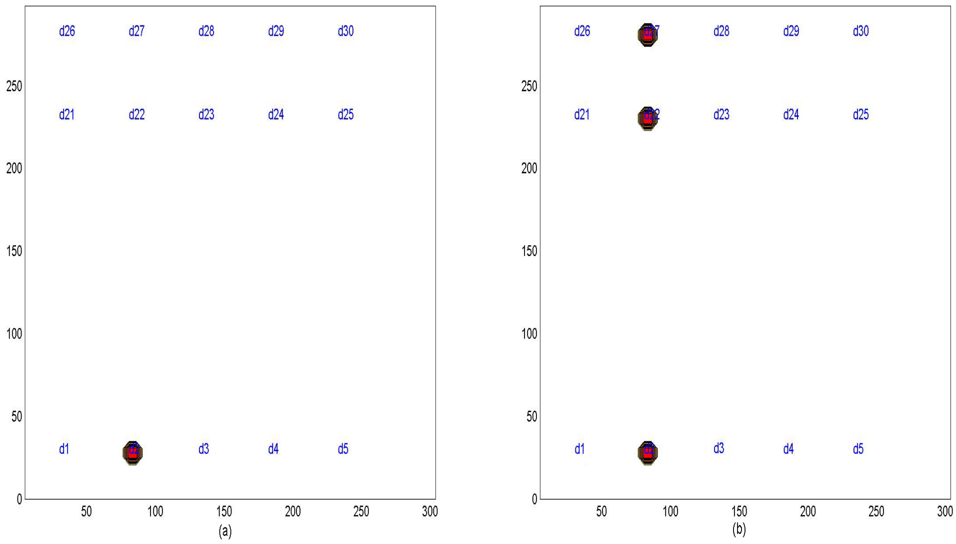


FIG. 7 – (a) : émergence sur 7 jours, (b) : émergence sur 14 jours (test 1.3)

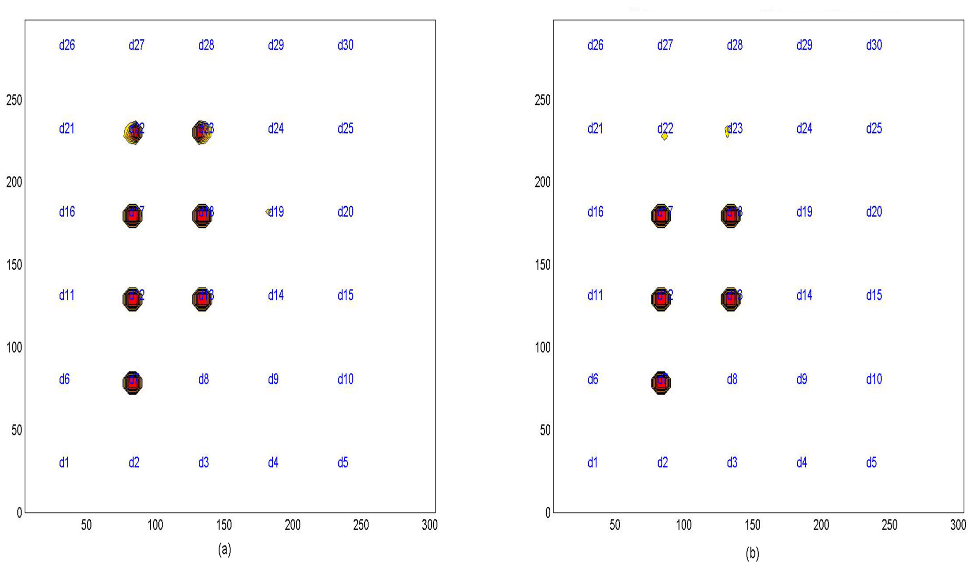


FIG. 8 – (a) : pas de diapause prolongée, (b) : diapause prolongée de 20% à 1 an, 30% à 2 ans (test 2.1)

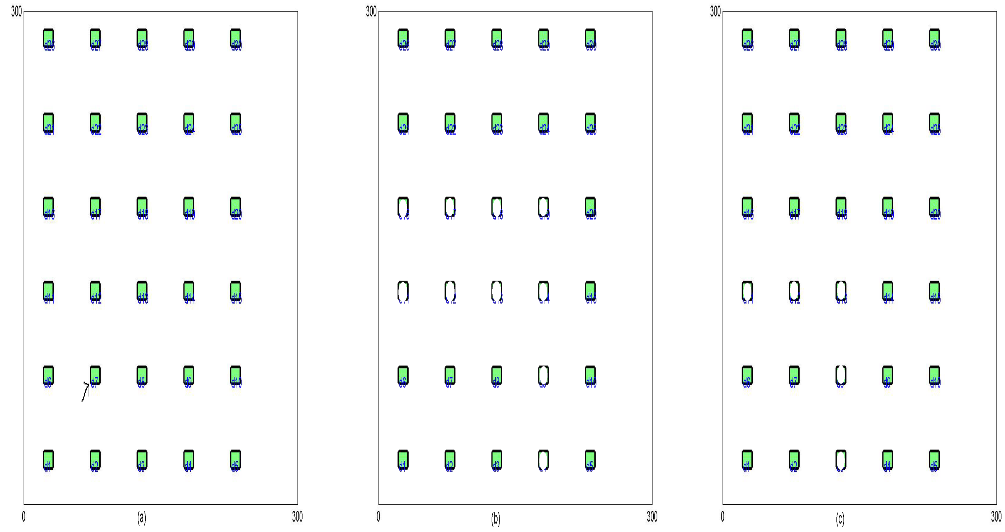


FIG. 9 – (a) : fructification la 1^{ere} année, (b) : fructification les années 2 et 3, (c) :fructification les années suivantes. Les parcelles marquées d'un rond blanc ont une fructification nulle. Ailleurs la fructification est de 10hl. (test 2.2)

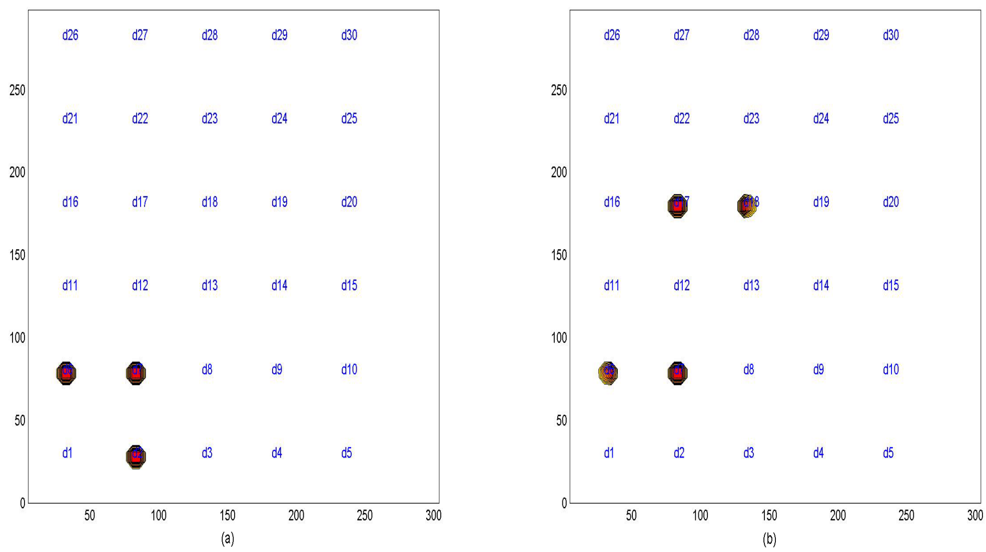


FIG. 10 – (a) : pas de diapause prolongée, (b) : diapause prolongée de 20% à 1 an, 30% à 2 ans (test 2.3)

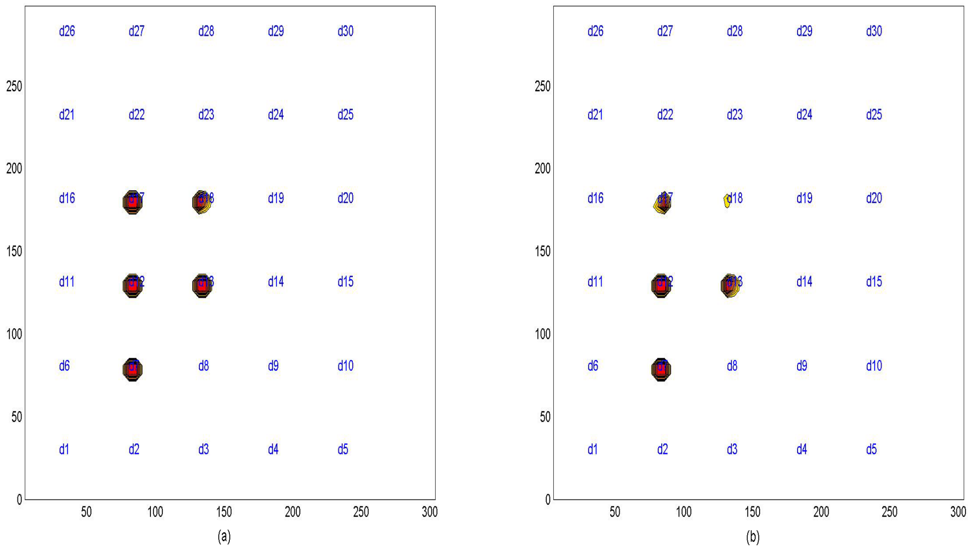


FIG. 11 – (a) : la fructification est constante d'année en année, (b) : la fructification est nulle la 2^{eme} année (test2.3)

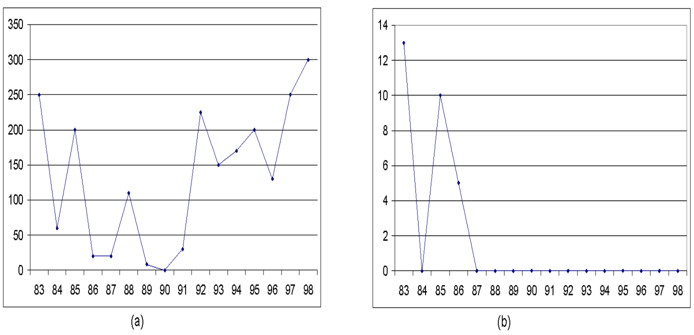


FIG. 12 – Fructification en *hl* de 1983 à 1998 sur (a) : la forêt du Ventoux, (b) : la forêt d'Oppede

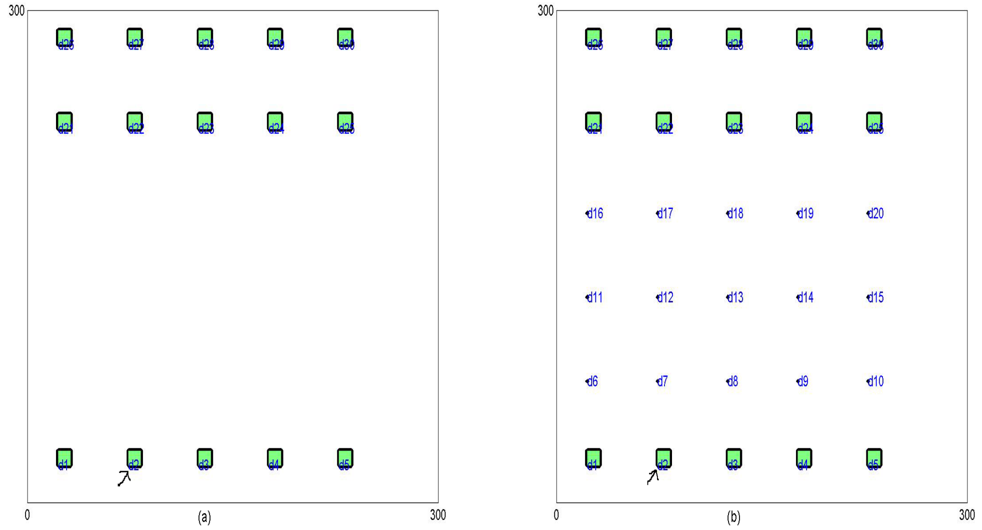


FIG. 13 – (a) : domaine sans arbre isolé, (b) : domaine avec arbres isolés (tests 3.1 et 3.2)

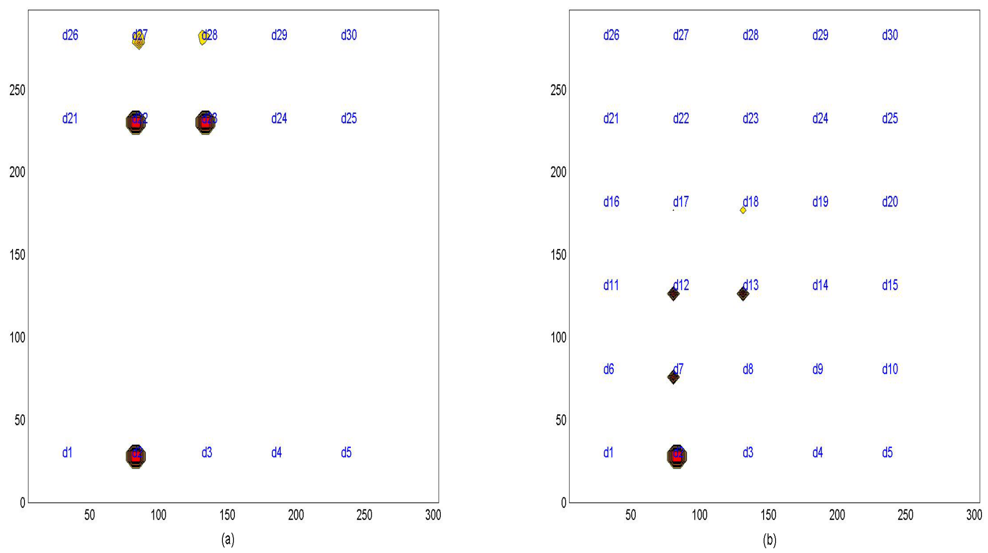


FIG. 14 – (a) : propagation sans arbre isolé, (b) : avec arbres isolés (test 3.1))

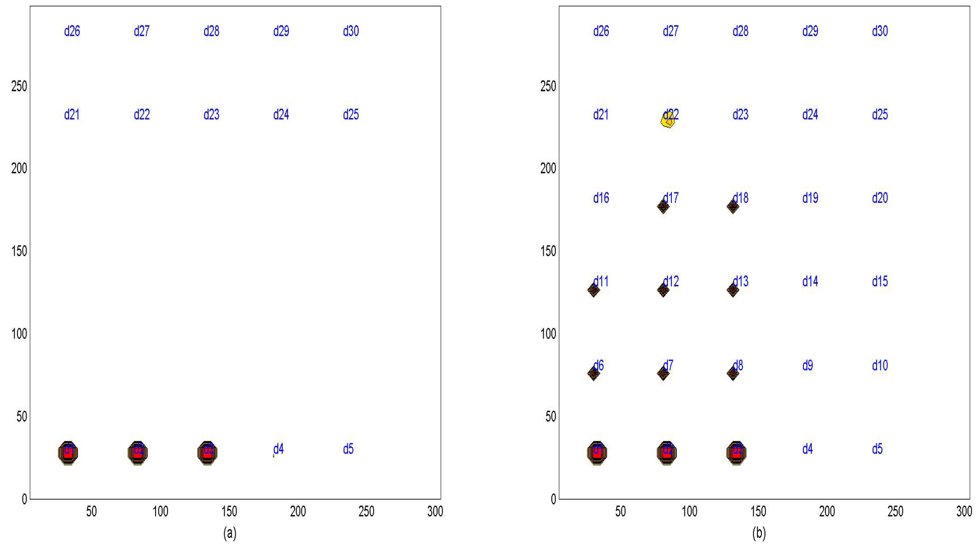


FIG. 15 – (a) : propagation sans arbre isolé, (b) : avec arbres isolés (test 3.2)

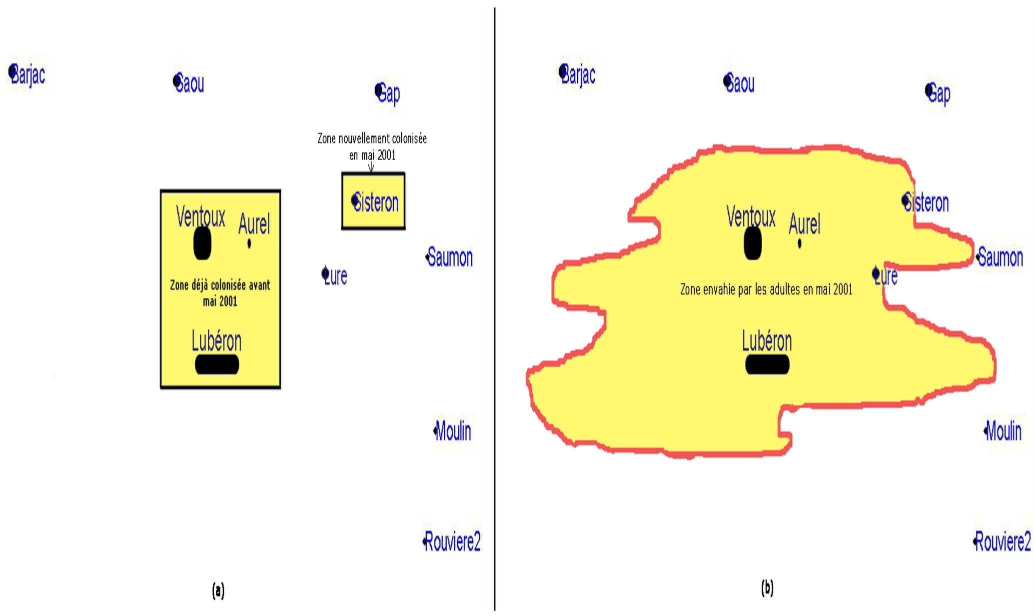


FIG. 16 – (a) : estimation expérimentale de l'invasion en mai 2001, (b) : invasion simulée par le modèle

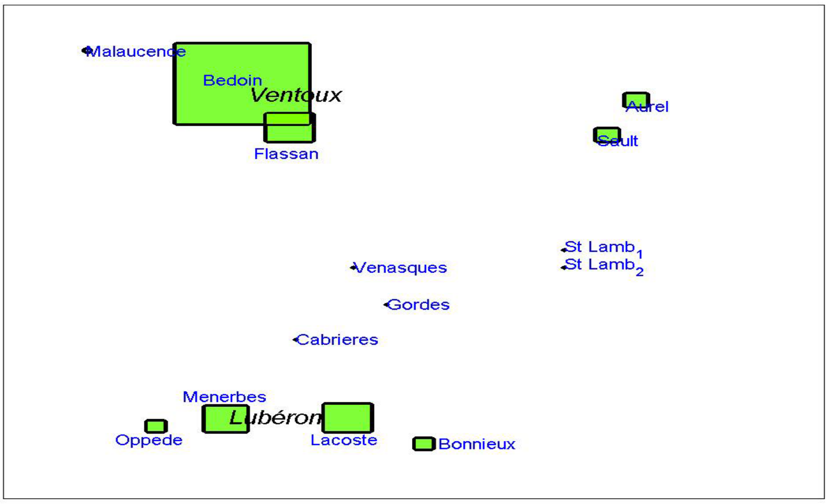


FIG. 17 – Carte géographique numérisée, et utilisée pour la validation du modèle

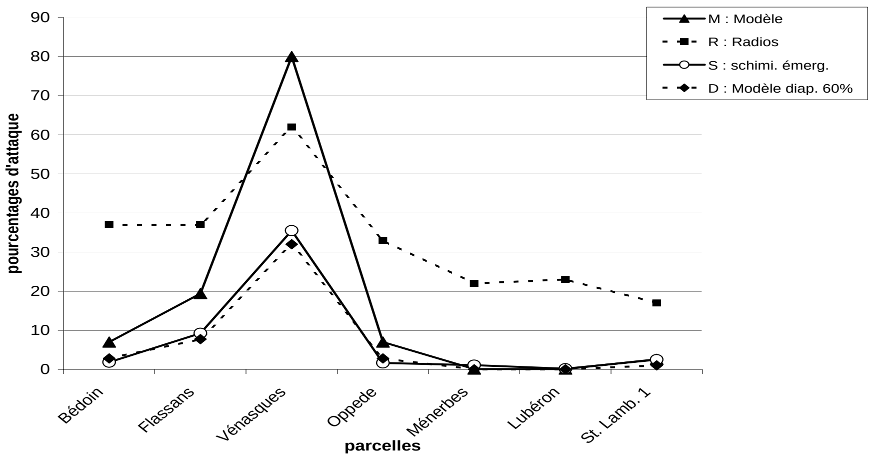


FIG. 18 – Comparaison entre le taux d'attaque prévu par le modèle, le taux d'attaque enregistré par radiographie, le taux d'émergence de *M. schimitscheki*, et le taux prévu par ce même modèle avec une diapause prolongée de 60%

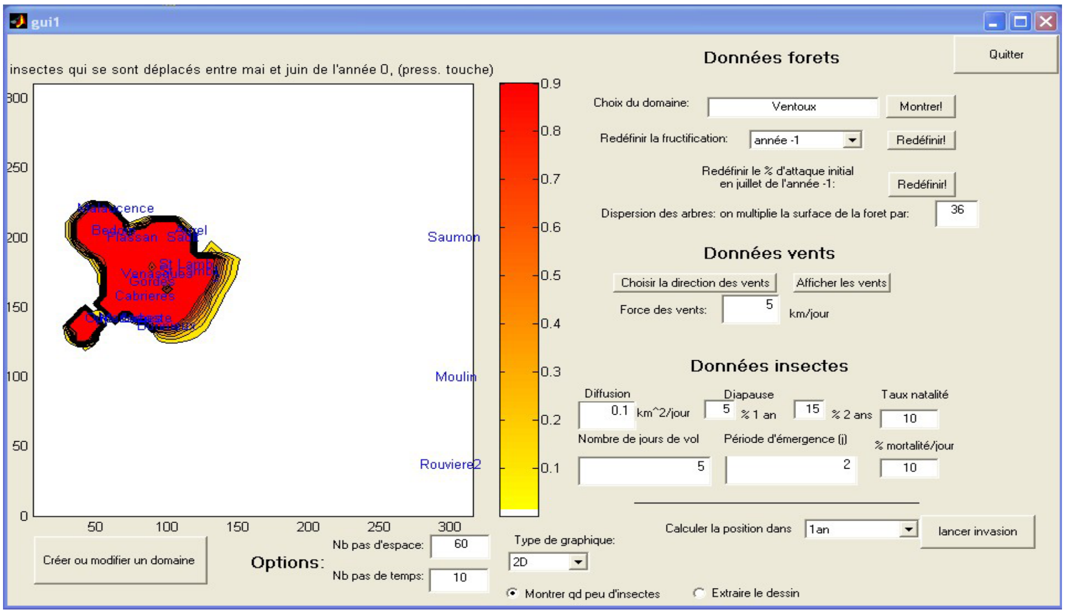


FIG. 19 – Interface graphique du programme permettant d'exploiter notre modèle

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